# Combining Functions

- Two functions $f$ and $g$ can be combined to form new functions
  - $f+g$,
  - $f-g$,
  - $fg$, and
  - $f/g$

  just as we add, subtract, multiply, and divide real numbers.

- This is summarized in the following table:

**Algebra of Functions** Let $f$ and $g$ be functions with domains $A$ and $B$. Then the functions $f+g$, $f-g$, $fg$, and $f/g$ are defined as follows:

$$(f+g)(x) = f(x) + g(x) \qquad \text{domain} = A \cap B$$

$$(f-g)(x) = f(x) - g(x) \qquad \text{domain} = A \cap B$$

$$(fg)(x) = f(x)g(x) \qquad \text{domain} = A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{domain} = \{x \in A \cap B \mid g(x) \neq 0\}$$

# Example

- If $f(x)=\sqrt{x}$ and $g(x)=\sqrt{4-x^2}$, find the functions $f+g$, $f-g$, $fg$, and $f/g$.

**Also examine the domain of each of these new functions

$f(x)=\sqrt{x}$

$x \geq 0$

$D: \{x \mid x \geq 0, x \in R\}$

$g(x)=\sqrt{4-x^2}$

$4-x^2 \geq 0$

$(2-x)(2+x)=0$

− + −

(-3) -2 (0) 2 (3)

$D: \{x \mid -2 \leq x \leq 2, x \in R\}$

The domain common to both $f(x)$ and $g(x)$ is $\{x \mid 0 \leq x \leq 2, x \in R\}$

① $f(x)+g(x)$

$= \sqrt{x} + \sqrt{4-x^2}$

$D: \{x \mid 0 \leq x \leq 2, x \in R\}$

② $f(x)-g(x)$

$= \sqrt{x} - \sqrt{4-x^2}$

$D: \{x \mid 0 \leq x \leq 2, x \in R\}$

③ $f(x) \cdot g(x)$

$= \sqrt{x} \cdot \sqrt{4-x^2}$

$= \sqrt{x(4-x^2)}$

$= \sqrt{4x-x^3}$

$D: \{x \mid 0 \leq x \leq 2, x \in R\}$

④ $\dfrac{f(x)}{g(x)} = \dfrac{\sqrt{x}}{\sqrt{4-x^2}} = \dfrac{\sqrt{4x-x^3}}{4-x^2}$

$D: \{x \mid 0 \leq x < 2, x \in R\}$

$x \neq \pm 2$

# Compositions of Functions

(combining functions in a different way)

When the input in a function is another function, the result is called a **composite function.** If

$f(x) = 3x + 2$ and $g(x) = 4x - 5$

then $f[g(x)]$ is a composite function. The statement $f[g(x)]$ is read " $f$ of $g$ of $x$" or "the composition of $f$ with $g$." $f[g(x)]$ can also be written as

$(f \circ g)(x)$ or $f \circ g(x)$

The symbol between $f$ and $g$ is a small open circle. When replacing one function with another, be very careful to get the order correct because compositions of functions are not necessarily commutative (as you'll see).

① $(f \circ g)(x) = f(g(x)) = f(4x-5) = 3(4x-5)+2$
$= 12x - 15 + 2$
$= 12x - 13$

② $g(f(x)) = g(3x+2) = 4(3x+2)-5$
$= 12x+8-5$
$= 12x+3$

## Example 1

### Evaluate a Composite Function

If $f(x) = 4x$, $g(x) = x + 6$, and $h(x) = x^2$, determine each value.

**a)** $f(g(3))$
**b)** $g(h(-2))$
**c)** $h(h(2))$

**Method 1: Determine the Value of the Inner Function and Then Substitute**

a) $g(x) = x + 6$
$g(3) = 3 + 6$
$g(3) = 9$

$f(x) = 4x$
$f(9) = 4(9)$
$f(9) = 36$

b) $h(x) = x^2$
$h(-2) = (-2)^2$
$h(-2) = 4$

$g(x) = x + 6$
$g(4) = (4) + 6$
$g(4) = 10$

c) $h(x) = x^2$
$h(2) = (2)^2$
$h(2) = 4$

$h(x) = x^2$
$h(4) = (4)^2$
$h(4) = 16$

**Example 2**

If $f(x) = 3x^2 + 2x + 1$ and $g(x) = 4x - 5$, find each of the following:

1. $f[g(x)]$
2. $g[f(x)]$

① $f(4x-5) = 3(4x-5)^2 + 2(4x-5) + 1$

$= 3(4x-5)(4x-5) + 8x - 10 + 1$

$= 3(16x^2 - 20x - 20x + 25) + 8x - 9$

$= 3(16x^2 - 40x + 25) + 8x - 9$

$= 48x^2 - 120x + 75 + 8x - 9$

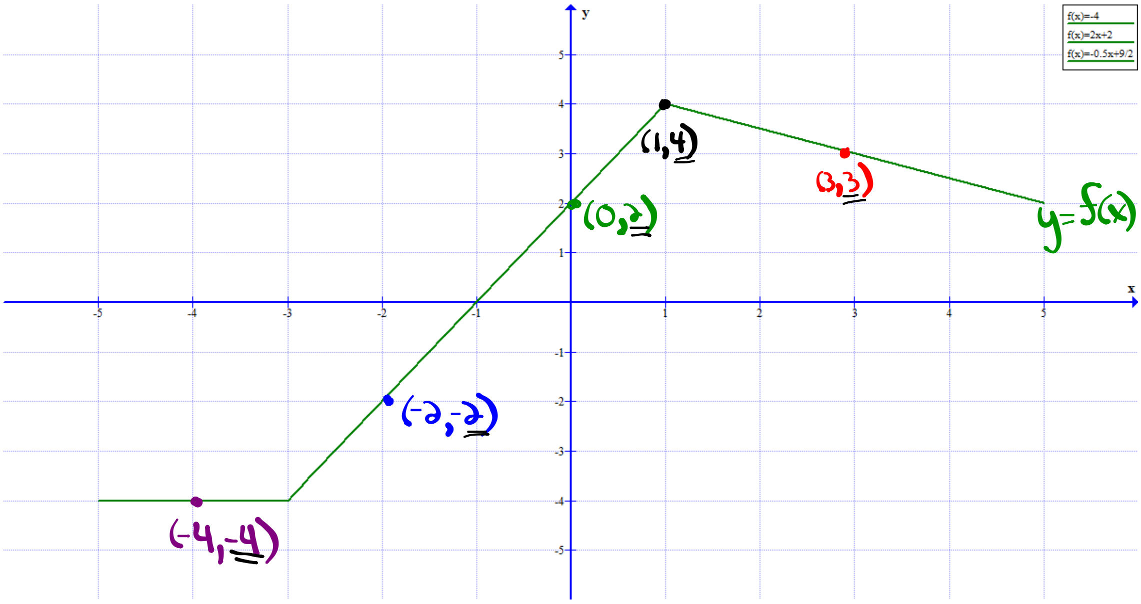
$\boxed{= 48x^2 - 112x + 66}$

② $g(3x^2+2x+1) = 4(3x^2+2x+1) - 5$

$= 12x^2 + 8x + 4 - 5$

$\boxed{= 12x^2 + 8x - 1}$

Given the graph of f(x) shown below, evaluate the following:




$$\frac{3f(1)-5\left[f(3)-7f(0)\right]}{2f(-2)-3f(-4)}$$

$$\frac{3(4)-5[3-7(2)]}{2(-2)-3(-4)}$$

$$\frac{12-5[3-14]}{-4+12}$$

$$\frac{12-5(-11)}{8}$$

$$\frac{12+55}{8}$$

$$\frac{77}{8}$$

## Key Ideas

- Two functions, $f(x)$ and $g(x)$, can be combined using composition to produce two new functions, $f(g(x))$ and $g(f(x))$.
- To evaluate a composite function, $f(g(x))$, at a specific value, substitute the value into the equation for $g(x)$ and then substitute the result into $f(x)$ and evaluate, or determine the composite function first and then evaluate for the value of $x$.
- To determine the equation of a composite function, substitute the second function into the first as read from left to right. To compose $f(g(x))$, substitute the equation of $g(x)$ into the equation of $f(x)$.
- The domain of $f(g(x))$ is the set of all values of $x$ in the domain of $g$ for which $g(x)$ is in the domain of $f$. Restrictions on the inner function as well as the composite function must be considered.

# Homework

## #1-10 on page 507 (omit #6)

## 10.3 Composite Functions, pages 507 to 509

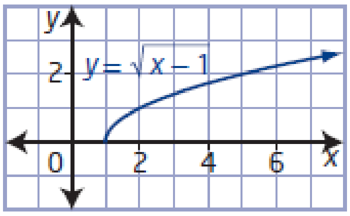
**1. a)** 3 **b)** 0 **c)** 2 **d)** $-1$

**2. a)** 2 **b)** 2 **c)** $-4$ **d)** $-5$

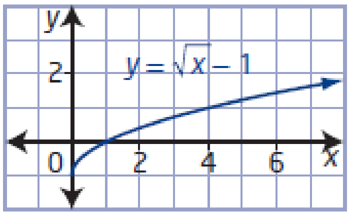
**3. a)** 10 **b)** $-8$ **c)** $-2$ **d)** 28

**4. a)** $f(g(a)) = 3a^2 + 1$ **b)** $g(f(a)) = 9a^2 + 24a + 15$
**c)** $f(g(x)) = 3x^2 + 1$ **d)** $g(f(x)) = 9x^2 + 24x + 15$
**e)** $f(f(x)) = 9x + 16$ **f)** $g(g(x)) = x^4 - 2x^2$

**5. a)** $f(g(x)) = x^4 + 2x^3 + 2x^2 + x$,
$g(f(x)) = x^4 + 2x^3 + 2x^2 + x$
**b)** $f(g(x)) = \sqrt{x^4 + 2}$, $g(f(x)) = x^2 + 2$
**c)** $f(g(x)) = x^2$, $g(f(x)) = x^2$

**6. a)**


domain
$\{x \mid x \geq 1, x \in \mathrm{R}\}$,
range
$\{y \mid y \geq 0, y \in \mathrm{R}\}$

**b)**


domain
$\{x \mid x \geq 0, x \in \mathrm{R}\}$,
range
$\{y \mid y \geq -1, y \in \mathrm{R}\}$

**7. a)** $g(x) = 2x - 5$ **b)** $g(x) = 5x + 1$

**8.** Christine is right. Ron forgot to replace all $x$'s with the other function in the first step.

**9.** Yes. $k(j(x)) = j(k(x)) = x^6$; using the power law: $2(3) = 6$ and $3(2) = 6$.

**10.** No. $s(t(x)) = x^2 - 6x + 10$ and $t(s(x)) = x^2 - 2$.

**11. a)** $W(C(t)) = 3\sqrt{100 + 35t}$
**b)** domain $\{t \mid t \geq 0, t \in \mathrm{R}\}$, range $\{W \mid W \geq 30, W \in \mathrm{W}\}$