Exam Outline:

- Limit definition of the derivative
- Derivatives (Do not Simplify / Simplify)
- Limits
- Curve Sketch
- Related Rate Problem
- Optimization Problem
- Riemann Sum (#6 Integration)
- Integrals (Indefinite / Definite)
- Area Between Curves. (#9 Integration)

① i) $y = x^x$

$\ln y = \ln x^{x}$

$\ln y = (x)(\ln x)$ ← product rule

$y \cdot \quad \frac{y'}{y} = \left[(1)(\ln x) + x\left(\frac{1}{x}\right)\right] \cdot y$

$y' = (\ln x + 1)\, x^x = x^x(\ln x + 1)$

③ $y = \frac{(3x+4)^8 \sqrt{2x^3-5}}{x^{14}} = \frac{(3x+4)^8 (2x^3-5)^{1/2}}{x^{14}}$

$\ln y = \ln\left(\frac{(3x+4)^8 (2x^3-5)^{1/2}}{x^{14}}\right)$

$\ln y = \ln(3x+4)^{8} + \ln(2x^3-5)^{1/2} - \ln x^{14}$

$\ln y = 8\ln(3x+4) + \frac{1}{2}\ln(2x^3-5) - 14\ln x$

$\frac{y'}{y} = 8\left(\frac{3}{3x+4}\right) + \frac{1}{2}\left(\frac{6x^2}{2x^3-5}\right) - 14\left(\frac{1}{x}\right)$

$y \cdot \quad \frac{y'}{y} = \left[\frac{24}{3x+4} + \frac{3x^2}{2x^3-5} - \frac{14}{x}\right] y$

$y' = \left[\frac{24}{3x+4} + \frac{3x^2}{2x^3-5} - \frac{14}{x}\right]\left[\frac{(3x+4)^8\sqrt{2x^3-5}}{x^{14}}\right]$

② $x^2+3xy=y^2+2x+1$

a) $x=-2$, $y=1$

$(-2)^2+3(-2)(1)=(1)^2+2(-2)+1$

$4-6 \quad | \quad 1-4+1$

$-2 \quad | \quad -2$ ✓

b) $x^2+(3x)(y)=y^2+2x+1$

$2x+3y+3x\frac{dy}{dx}=2y\frac{dy}{dx}+2$

$2x+3y-2=2y\frac{dy}{dx}-3x\frac{dy}{dx}$

$2x+3y-2=\frac{dy}{dx}(2y-3x)$

$$\boxed{\frac{2x+3y-2}{2y-3x}=\frac{dy}{dx}} \leftarrow m$$

c) ① Find m:

$m=\frac{dy}{dx}=\frac{2(-2)+3(1)-2}{2(1)-3(-2)}=\frac{-3}{8}$

② $y-y_1=m(x-x_1)$

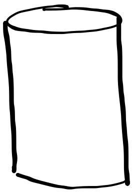
$y-1=\frac{-3}{8}(x+2)$

$8\cdot \quad 8\cdot y - 8\cdot 1=8\cdot\frac{-3x}{8}-8\cdot\frac{6}{8}$

$8y-8=-3x-6$

$$\boxed{3x+8y-2=0}$$

(10)



$V = \pi r^2 h$

$2000 = \pi r^2 h$

$\frac{2000}{\pi r^2} = h$

$A = 2\pi r^2 + 2\pi r h$

$A = 2\pi r^2 + 2\pi r\left(\frac{2000}{\pi r^2}\right)$

$A = 2\pi r^2 + \frac{4000\pi r}{\pi r^2}$

$A = 2\pi r^2 + 4000 r^{-1}$

$A' = 4\pi r - 4000 r^{-2}$

$A' = \frac{4\pi r}{1} - \frac{4000}{r^2}$

$A' = \frac{4\pi r^3 - 4000}{r^2}$

$A' = \frac{4\pi r^3 - 4000}{r^2}$

$0 = 4\pi r - \frac{4000}{r^2}$

$\frac{4000}{r^2} = 4\pi r$

$4\pi r^3 = 4000$

$r^3 = \frac{1000}{\pi}$

$r = \frac{10}{\sqrt[3]{\pi}} \doteq 9.55\text{ cm}$

Critical Values

$4\pi r^3 - 4000 = 0$

$\frac{4\pi r^3}{4\pi} = \frac{4000}{4\pi}$

$r^3 = \frac{1000}{\pi}$

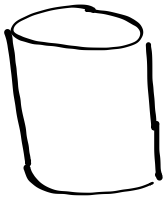
$r = \frac{\sqrt[3]{1000}}{\sqrt[3]{\pi}}$

$r = \frac{10}{\sqrt[3]{\pi}}$ min

$r^2 = 0$

$r = 0$

(11)



$A = 2\pi r^2 + 2\pi r h$

$2400 = 2\pi r^2 + 2\pi r h$

$2400 - 2\pi r^2 = 2\pi r h$

$\frac{2400 - 2\pi r^2}{2\pi r} = h$

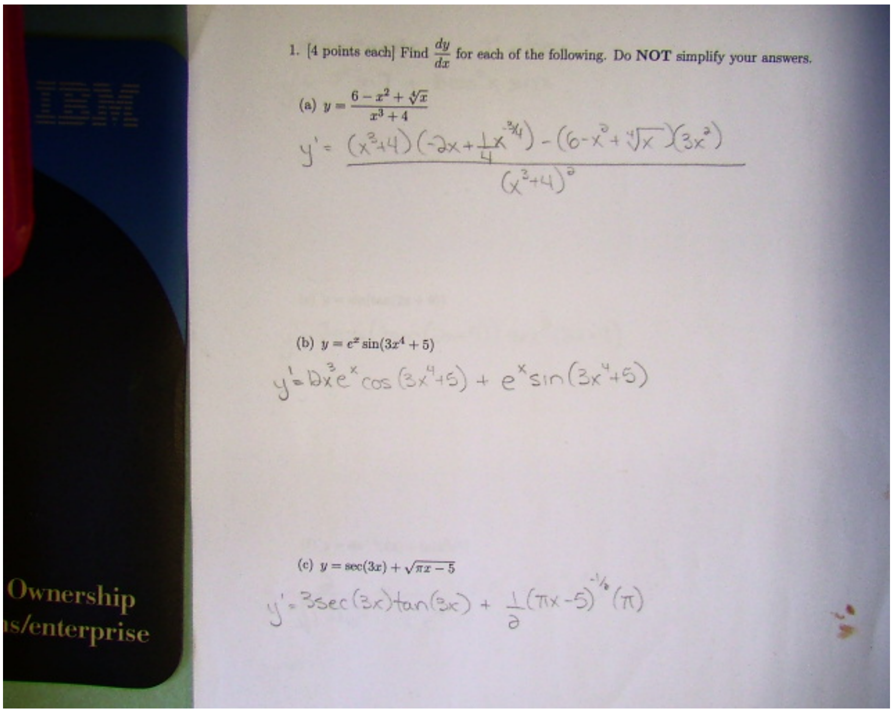
$\boxed{\frac{1200 - \pi r^2}{\pi r}} = h$

$V = \pi r^2 h$

$V = \pi r^2 \left(\frac{1200 - \pi r^2}{\pi r}\right)$

$V = 1200r - \pi r^3$

↓



1. [4 points each] Find $\frac{dy}{dx}$ for each of the following. Do **NOT** simplify your answers.

(a) $y = \frac{6 - x^2 + \sqrt[4]{x}}{x^3 + 4}$

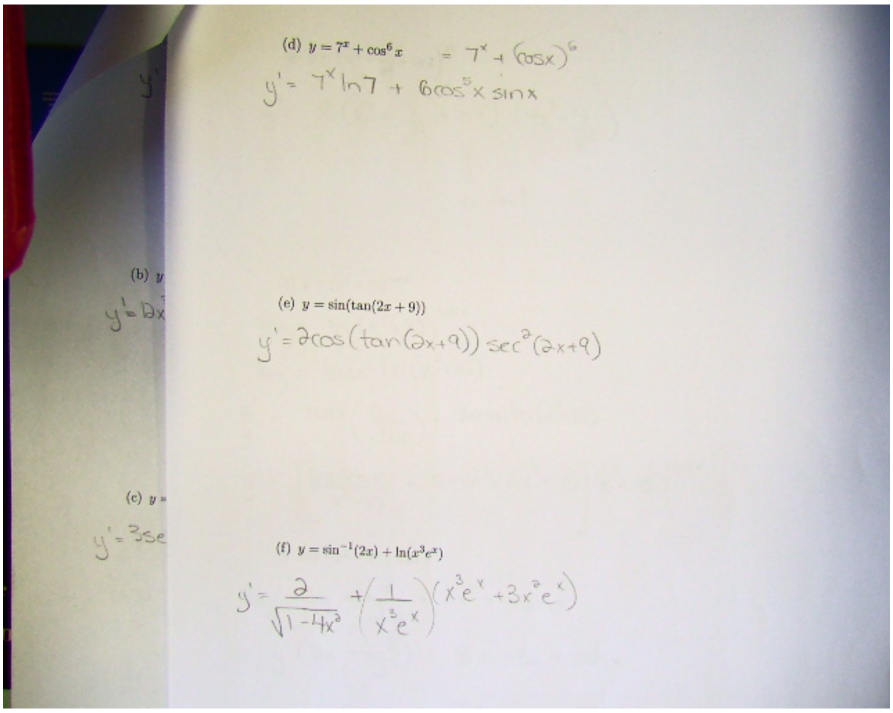
$y' = \frac{(x^3+4)(-2x + \frac{1}{4}x^{-3/4}) - (6 - x^2 + \sqrt[4]{x})(3x^2)}{(x^3+4)^2}$

(b) $y = e^x \sin(3x^4 + 5)$

$y' = 12x^3 e^x \cos(3x^4+5) + e^x \sin(3x^4+5)$

(c) $y = \sec(3x) + \sqrt{\pi x - 5}$

$y' = 3\sec(3x)\tan(3x) + \frac{1}{2}(\pi x - 5)^{-1/2}(\pi)$



(d) $y = 7^x + \cos^6 x$ $= 7^x + (\cos x)^6$

$y' = 7^x \ln 7 + 6\cos^5 x \sin x$

(b) y

$y' = 2x$

(e) $y = \sin(\tan(2x+9))$

$y' = 2\cos(\tan(2x+9))\sec^2(2x+9)$

(c) $y =$

$y' = 3se$

(f) $y = \sin^{-1}(2x) + \ln(x^3e^x)$

$y' = \frac{2}{\sqrt{1-4x^2}} + \left(\frac{1}{x^3e^x}\right)(x^3e^x + 3x^2e^x)$

(g) $y = \left(x^4 + \frac{1}{x^4} + \ln 4\right)^{15}$

$-4x^{-5}$

$y' = 15\left(x^4 + \frac{1}{x^4} + \ln 4\right)^{15}\left(4x^3 - \frac{4}{x^5}\right)$

↑ constant

(h) $y = (x^2+10)^{\cos x}$

$\ln y = \ln (x^2+10)^{\cos x}$

$\ln y = \cos x \ln(x^2+10)$

$\frac{y'}{y} = \cos x\left(\frac{2x}{x^2+10}\right) + \sin x \ln(x^2+10)$

$y' = \left[\frac{2x\cos x}{x^2+10} + \sin x \ln(x^2+10)\right](x^2+10)^{\cos x}$

(i) $3xy + y^4 = x^8 + \sinh x$

$3xy' + 3y + 4y^3y' = 8x^7 + \cosh x$

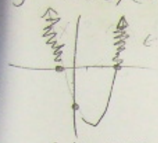
$y'(3x + 4y^3) = 8x^7 - 3y + \cosh x$

$y' = \frac{8x^7 - 3y + \cosh x}{(3x + 4y^3)}$

2. [4 points] Find the domain of $f(x) = \sqrt{x^2 - 2x - 3}$

$x^2 - 2x - 3 \geq 0$ ← cannot take the square root of a negative

$y = (x-3)(x+1)$



D: $\{x \mid x \leq -1 \text{ and } x \geq 3, x \in \mathbb{R}\}$

3. [3 points] Answer (a) and (b) for the function f defined below.

$$f(x) = \begin{cases} x - 1 & \text{if } x < 1; \\ (x-1)^2 & \text{if } x > 1; \\ 3 & \text{if } x = 1. \end{cases}$$

$x-1$

x	f(x)
1	0
0	-1
-1	-2

$(x-1)^2$

x	f(x)
1	0
2	1
3	4

3

x	f(x)
1	3

(a) Find $\lim_{x \to 1} f(x)$.

$\lim_{x \to 1^-} f(x)$

$\lim_{x \to 1^-} x - 1 = 0$

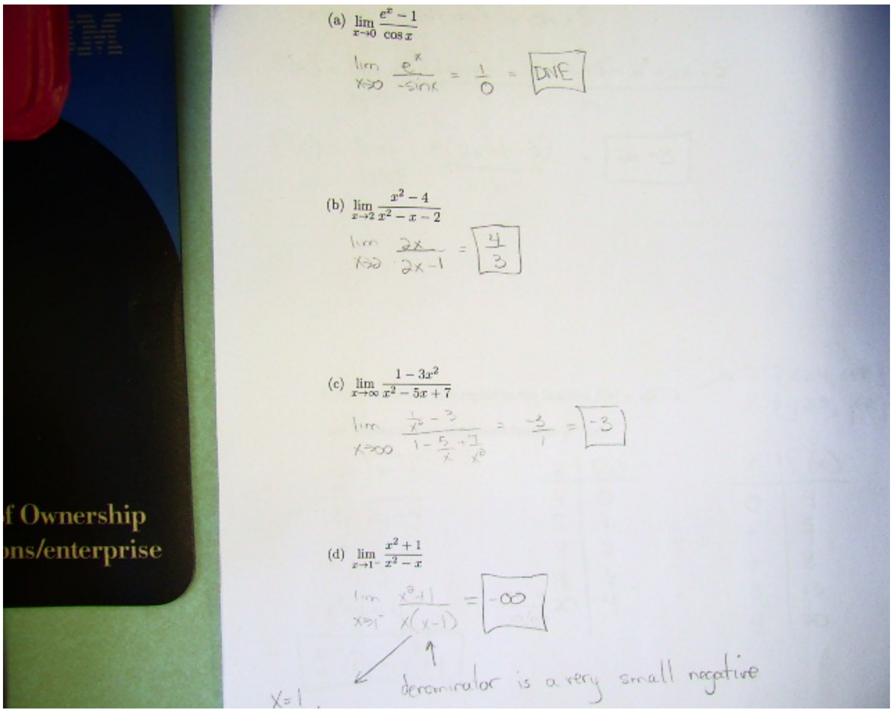
$\lim_{x \to 1^+} f(x)$

$\lim_{x \to 1^+} (x-1)^2 = 0$

$\lim_{x \to 1} f(x) = 0$

(b) Is $f(x)$ continuous at $x = 1$? Justify your answer.

No because $\lim_{x \to 1} f(x) = 0$ and $f(1) = 3$



(a) $\lim_{x\to 0} \frac{e^x - 1}{\cos x}$

$\lim_{x\to 0} \frac{e^x}{-\sin x} = \frac{1}{0} = \boxed{\text{DNE}}$

(b) $\lim_{x\to 2} \frac{x^2 - 4}{x^2 - x - 2}$

$\lim_{x\to 2} \frac{2x}{2x-1} = \boxed{\frac{4}{3}}$

(c) $\lim_{x\to\infty} \frac{1 - 3x^2}{x^2 - 5x + 7}$

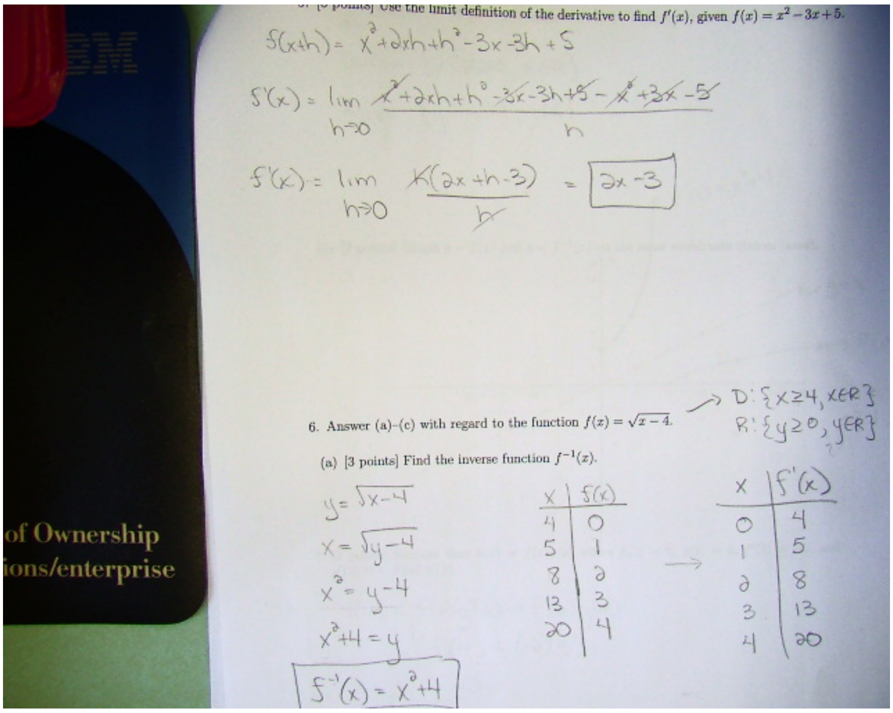
$\lim_{x\to\infty} \frac{\frac{1}{x^2} - 3}{1 - \frac{5}{x} + \frac{7}{x^2}} = \frac{-3}{1} = \boxed{-3}$

(d) $\lim_{x\to 1^-} \frac{x^2 + 1}{x^2 - x}$

$\lim_{x\to 1^-} \frac{x^2+1}{x(x-1)} = \boxed{-\infty}$

denominator is a very small negative

x=1



5. [5 points] Use the limit definition of the derivative to find $f'(x)$, given $f(x) = x^2 - 3x + 5$.

$f(x+h) = x^2 + 2xh + h^2 - 3x - 3h + 5$

$f'(x) = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h + 5 - x^2 + 3x - 5}{h}$

$f'(x) = \lim_{h \to 0} \frac{h(2x + h - 3)}{h} = \boxed{2x - 3}$

6. Answer (a)–(c) with regard to the function $f(x) = \sqrt{x - 4}$. → D: $\{x \geq 4, x \in R\}$ R: $\{y \geq 0, y \in R\}$

(a) [3 points] Find the inverse function $f^{-1}(x)$.

$y = \sqrt{x-4}$

$x = \sqrt{y-4}$

$x^2 = y - 4$

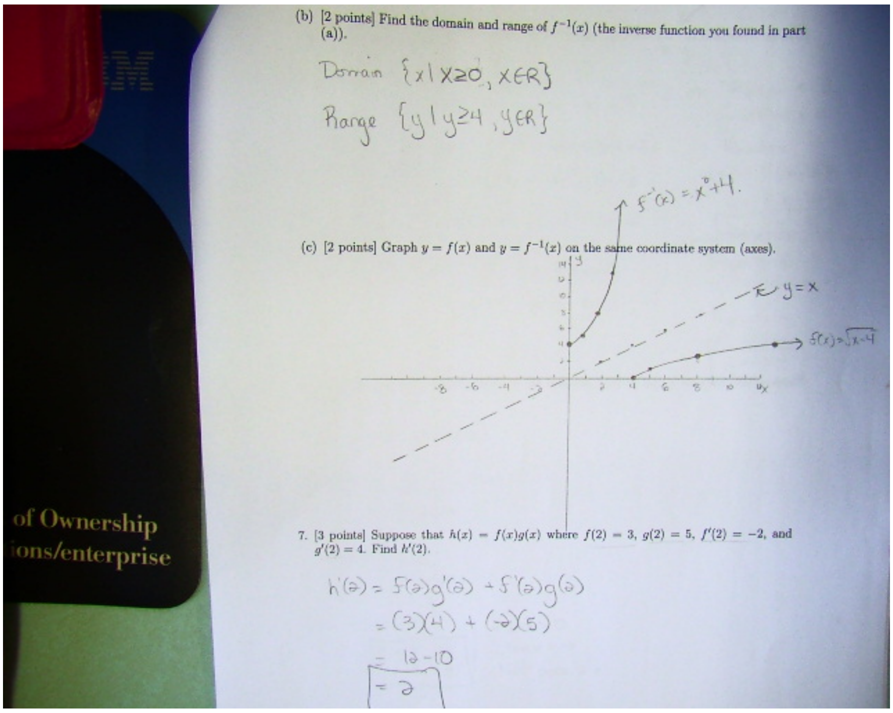
$x^2 + 4 = y$

$\boxed{f^{-1}(x) = x^2 + 4}$

x	f(x)
4	0
5	1
8	2
13	3
20	4

→

x	$f^{-1}(x)$
0	4
1	5
2	8
3	13
4	20



(b) [2 points] Find the domain and range of $f^{-1}(x)$ (the inverse function you found in part (a)).

Domain $\{x \mid x \geq 0, x \in R\}$

Range $\{y \mid y \geq 4, y \in R\}$

(c) [2 points] Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system (axes).

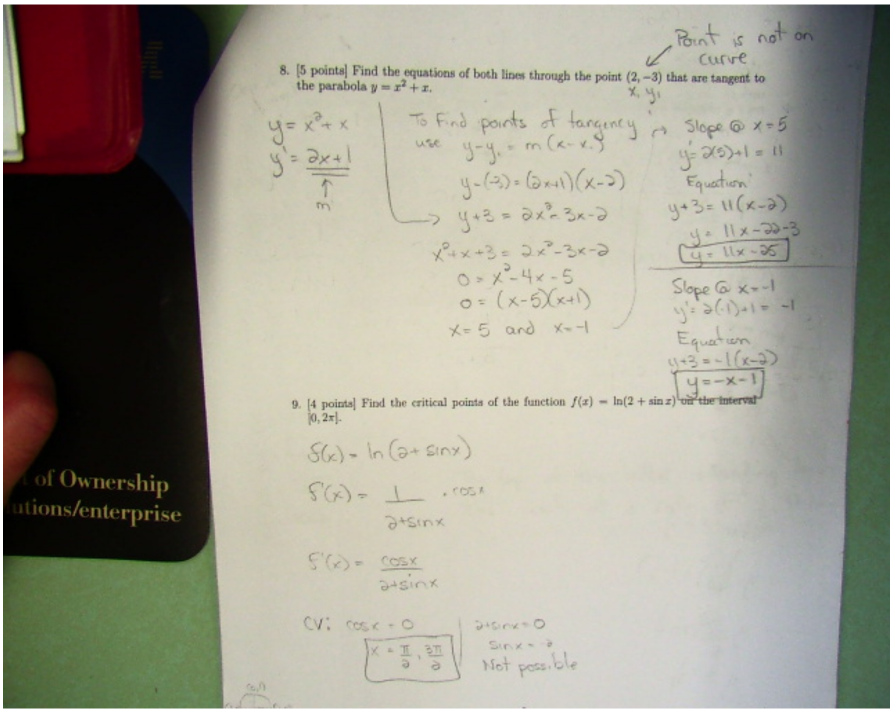
7. [3 points] Suppose that $h(x) = f(x)g(x)$ where $f(2) = 3$, $g(2) = 5$, $f'(2) = -2$, and $g'(2) = 4$. Find $h'(2)$.

$h'(2) = f(2)g'(2) + f'(2)g(2)$

$= (3)(4) + (-2)(5)$

$= 12 - 10$

$= 2$



8. [5 points] Find the equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.

Point is not on curve

x_1, y_1

$y = x^2 + x$

$y' = 2x + 1$ ← m

To find points of tangency use $y - y_1 = m(x - x_1)$

$y - (-3) = (2x+1)(x-2)$

$y + 3 = 2x^2 - 3x - 2$

$x^2 + x + 3 = 2x^2 - 3x - 2$

$0 = x^2 - 4x - 5$

$0 = (x-5)(x+1)$

$x = 5$ and $x = -1$

Slope @ $x = 5$

$y' = 2(5) + 1 = 11$

Equation

$y + 3 = 11(x-2)$

$y = 11x - 22 - 3$

$y = 11x - 25$

Slope @ $x = -1$

$y' = 2(-1) + 1 = -1$

Equation

$y + 3 = -1(x-2)$

$y = -x - 1$

9. [4 points] Find the critical points of the function $f(x) = \ln(2 + \sin x)$ ~~on the interval~~ $[0, 2\pi]$.

$f(x) = \ln(2 + \sin x)$

$f'(x) = \frac{1}{2 + \sin x} \cdot \cos x$

$f'(x) = \frac{\cos x}{2 + \sin x}$

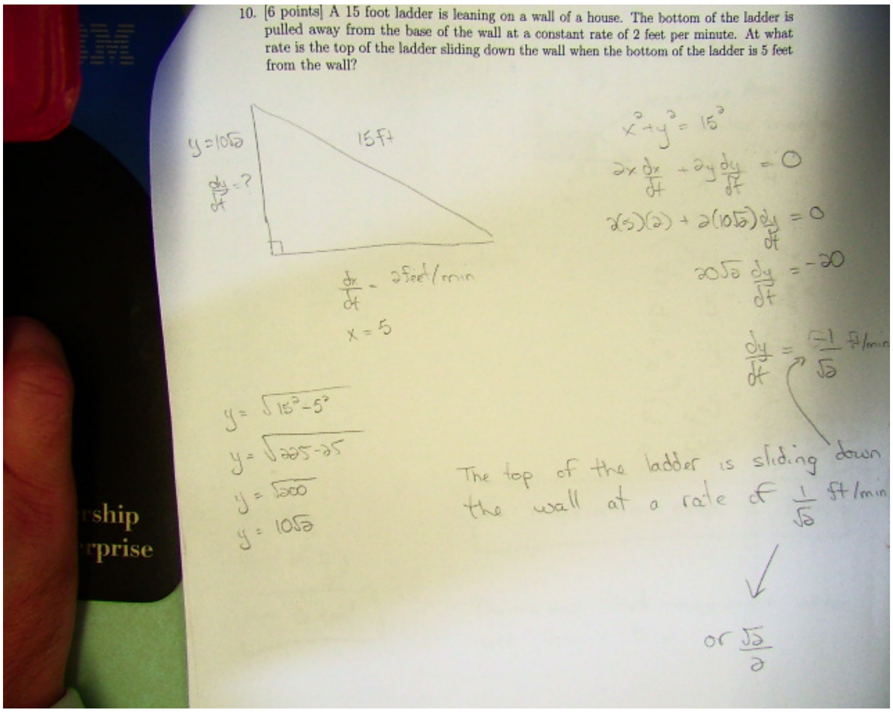
CV: $\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$2 + \sin x = 0$

$\sin x = -2$

Not possible



10. [6 points] A 15 foot ladder is leaning on a wall of a house. The bottom of the ladder is pulled away from the base of the wall at a constant rate of 2 feet per minute. At what rate is the top of the ladder sliding down the wall when the bottom of the ladder is 5 feet from the wall?

$y = 10\sqrt{2}$

$\frac{dy}{dt} = ?$

15 ft

$\frac{dx}{dt} = 2$ feet/min

$x = 5$

$y = \sqrt{15^2 - 5^2}$

$y = \sqrt{225 - 25}$

$y = \sqrt{200}$

$y = 10\sqrt{2}$

$x^2 + y^2 = 15^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2(5)(2) + 2(10\sqrt{2})\frac{dy}{dt} = 0$

$20\sqrt{2}\frac{dy}{dt} = -20$

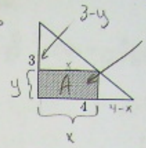
$\frac{dy}{dt} = \frac{-1}{\sqrt{2}}$ ft/min

The top of the ladder is sliding down the wall at a rate of $\frac{1}{\sqrt{2}}$ ft/min

or $\frac{\sqrt{2}}{2}$

11. [6 points] Find the area of the largest rectangle that can be inscribed in a right triangle with legs of 3cm and 4cm if two sides of the rectangle lie along the legs.

express with a single variable



maximize Area

$A = xy$

$A = x\left(3 - \frac{3}{4}x\right)$

$A = 3x - \frac{3}{4}x^2$

Differentiate

$A' = 3 - \frac{3}{2}x$

$\frac{3}{2}x = 3$

$3x = 6$

$\boxed{x = 2}$

similar triangles:

$\frac{3-y}{x} = \frac{3}{4}$

$3x = 12 - 4y$

$4y = 12 - 3x$

$y = 3 - \frac{3}{4}x$

$y = 3 - \frac{3}{4}(2)$

$y = 3 - \frac{3}{2}$

$y = \frac{6-3}{2} = \boxed{\frac{3}{2}}$

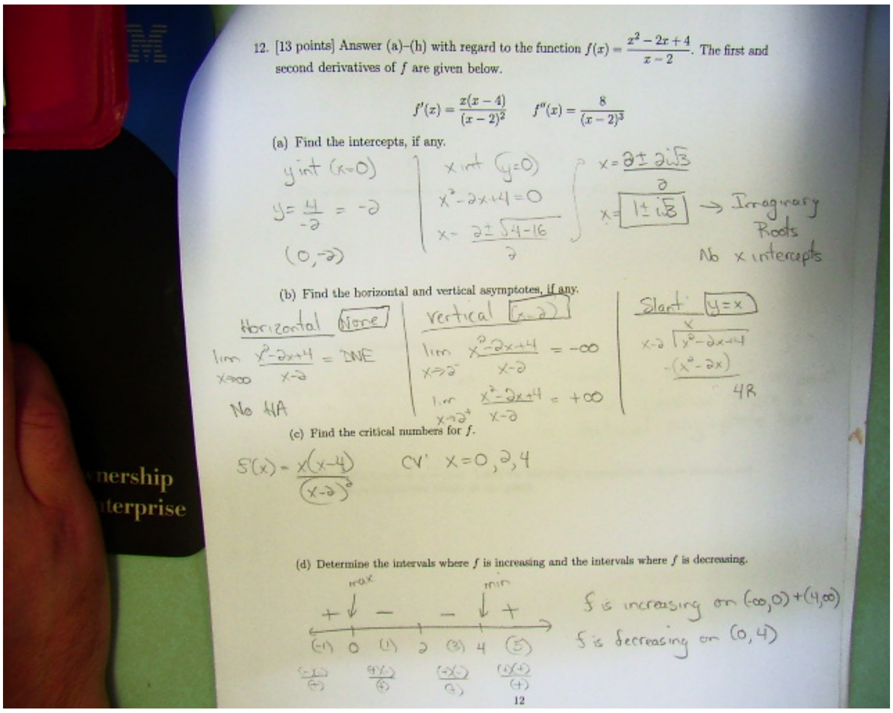
$A = xy$

$A = 2\left(\frac{3}{2}\right)$

$\boxed{A = 3\ cm^2}$

The maximum area is $3cm^2$.

Dimensions that maximize area are 2cm x 1.5cm



12. [13 points] Answer (a)–(h) with regard to the function $f(x) = \frac{x^2 - 2x + 4}{x - 2}$. The first and second derivatives of f are given below.

$$f'(x) = \frac{x(x-4)}{(x-2)^2} \qquad f''(x) = \frac{8}{(x-2)^3}$$

(a) Find the intercepts, if any.

y int (x=0)

$y = \frac{4}{-2} = -2$

(0, −2)

x int (y=0)

$x^2 - 2x + 4 = 0$

$x = \frac{2 \pm \sqrt{4-16}}{2}$

$x = \frac{2 \pm 2i\sqrt{3}}{2}$

$x = \boxed{1 \pm i\sqrt{3}}$ → Irragnary Roots

No x intercepts

(b) Find the horizontal and vertical asymptotes, if any.

Horizontal: None

$\lim_{x \to \infty} \frac{x^2 - 2x + 4}{x - 2} = \text{DNE}$

No HA

Vertical: $\boxed{x = 2}$

$\lim_{x \to 2^-} \frac{x^2 - 2x + 4}{x - 2} = -\infty$

$\lim_{x \to 2^+} \frac{x^2 - 2x + 4}{x - 2} = +\infty$

Slant: $\boxed{y = x}$

$x - 2 \overline{) x^2 - 2x + 4}$ quotient x

$-(x^2 - 2x)$

4R

(c) Find the critical numbers for f.

$f'(x) = \frac{x(x-4)}{(x-2)^2}$ CV: $x = 0, 2, 4$

(d) Determine the intervals where f is increasing and the intervals where f is decreasing.

max at 0, min at 4

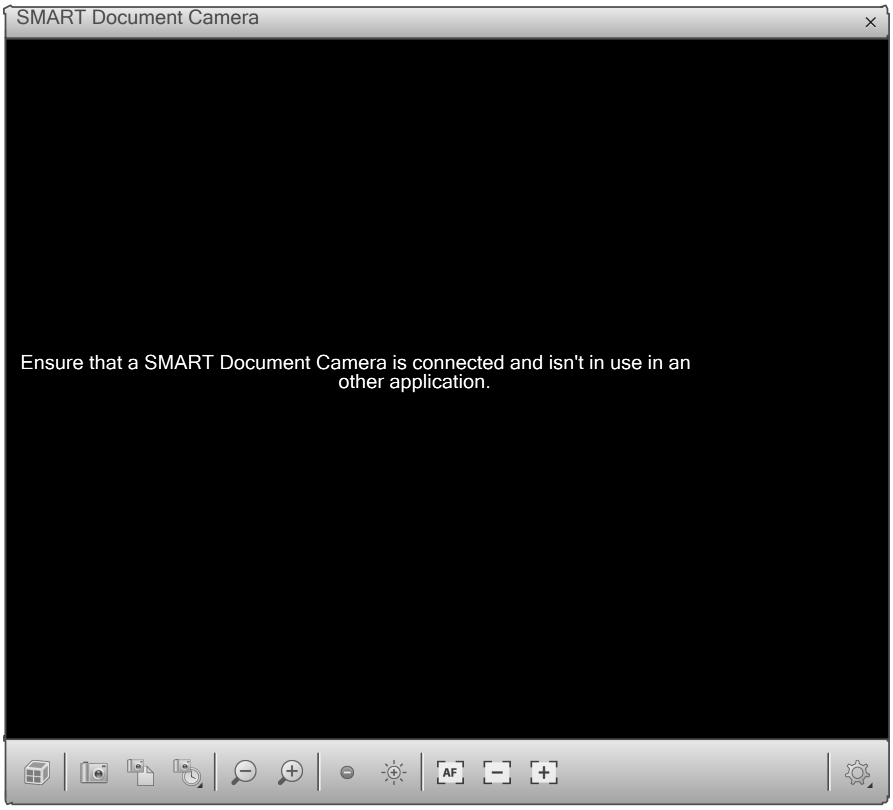
+ | − | − | +

(−1) 0 (1) 2 (3) 4 (5)

f is increasing on $(-\infty, 0) + (4, \infty)$

f is decreasing on $(0, 4)$

12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