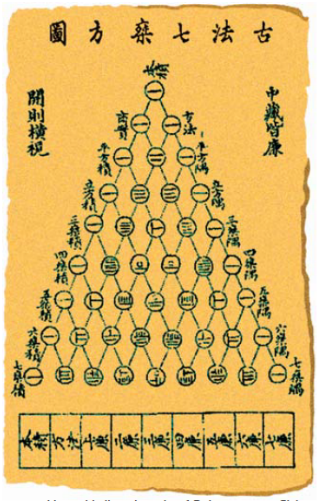
The Binomial Theorem

Focus on...

- relating the coefficients in the expansion of $(x + y)^n$, $n \in N$, to Pascal's triangle and to combinations
- expanding $(x + y)^n$, $n \in N$, in a variety of ways, including the binomial theorem
- determining a specific term in the expansion of $(x + y)^n$

In 1653, Blaise Pascal, a French mathematician, described a triangular array of numbers corresponding to the number of ways to choose *r* elements from a set of *n* objects. Some interesting number patterns occur in Pascal's triangle. Have you encountered Pascal's triangle before? Have you explored its many patterns? Did you realize it can give you the number of combinations in certain situations?



Yang Hui's triangle, 13th century China

Investigate Patterns in Pascal's Triangle

Materials
- counters
- copy of Pascal's triangle

1. Examine Pascal's triangle and identify at least three patterns. Compare and discuss your patterns with a partner.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

• symmetry
• each row starts and ends with 1
• second entry increases by 1

2. Write the next row for the Pascal's triangle shown. 1 5 10 10 5 1
3. Some of the patterns in Pascal's triangle are spatial and relate to whole sections in the chart. Create a large Pascal's triangle with at least 20 rows. Mark or use counters to cover all of the multiples of 7 in your 20-row triangle. Then, cover all of the multiples of 5 and multiples of 3. What do you conclude? What happens for multiples of even numbers? (On your own paper)
4. Other patterns may appear unexpectedly. Determine the sum of the numbers in each horizontal row. What pattern did you find?

the sums form a geometric sequene where the common ratio is 2

5. Each number in Pascal's triangle can be written as a combination using the notation ${}_nC_r$, where n is the number of objects in the set and r is the number selected. For example, you can express the third row as

${}_2C_0 \quad {}_2C_1 \quad {}_2C_2$

Express the fifth row using combination notation. Check whether your combinations have the same values as the numbers in the fifth row of Pascal's triangle. ${}_4C_0 \quad {}_4C_1 \quad {}_4C_2 \quad {}_4C_3 \quad {}_4C_4$

6. Expand the following binomials by multiplying.
$(x + y)^2$
$(x + y)^3$
$(x + y)^4$

Did You Know?

Pascal was not the first person to discover the triangle of numbers that bears his name. It was known in India, Persia, and China centuries before. The Chinese called it "Yang Hui's triangle" in honour of Yang Hui, who lived from 1238 to 1298.

Reflect and Respond

7. Explain how to get the numbers in the next row from the numbers in the previous row of Pascal's triangle. Use examples.
8. How are the values you obtained in steps 4 and 5 related? Explain using values from specific rows.
9. How do the coefficients of the simplified terms in your binomial expansions in step 6 relate to Pascal's triangle?

$(x+y)^2$
$(x+y)(x+y)$
$x^2+xy+xy+y^2$
$1x^2+2xy+1y^2$

$(x+y)^3$
$(x+y)(x+y)(x+y)$
$(x+y)(x^2+2xy+y^2)$
$x^3+2x^2y+xy^2+x^2y+2xy^2+y^3$
$1x^3+3x^2y+3xy^2+1y^3$

$(x+y)^4$
$1x^4y^0+4x^3y^1+6x^2y^2+4x^1y^3+1x^0y^4$
$x^4+4x^3y+6x^2y^2+4xy^3+y^4$

Link the Ideas

$(x+y)^6$
coefficients would be in row 7

If you expand a power of a binomial expression, you get a series of terms.

$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

How could you get this expansion by multiplying?

There are many patterns in the binomial expansion of $(x + y)^4$.

What patterns do you observe?

The coefficients in a binomial expansion can be determined from Pascal's triangle. In the expansion of $(x + y)^n$, where $n \in N$, the coefficients of the terms are identical to the numbers in the $(n + 1)$th row of Pascal's triangle.

Binomial	Pascal's Triangle in Binomial Expansion	Row
$(x+y)^0$	1	1
$(x+y)^1$	$1x + 1y$	2
$(x+y)^2$	$1x^2 + 2xy + 1y^2$	3
$(x+y)^3$	$1x^3 + 3x^2y + 3xy^2 + 1y^3$	4
$(x+y)^4$	$1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$	5

The coefficients in a binomial expansion can also be determined using combinations.

Pascal's Triangle	Combinations
1	$_0C_0$
1 1	$_1C_0$ $_1C_1$
1 2 1	$_2C_0$ $_2C_1$ $_2C_2$
1 3 3 1	$_3C_0$ $_3C_1$ $_3C_2$ $_3C_3$
1 4 6 4 1	$_4C_0$ $_4C_1$ $_4C_2$ $_4C_3$ $_4C_4$
1 5 10 10 5 1	$_5C_0$ $_5C_1$ $_5C_2$ $_5C_3$ $_5C_4$ $_5C_5$

$$
\begin{aligned}
{}_5C_2 &= \frac{5!}{3!2!} \\
&= \frac{(5)(4)}{2} \\
&= 10
\end{aligned}
$$

Note that $_5C_2$ represents the number of combinations of five items taken two at a time. In the expansion of $(x + y)^5$, it represents the coefficient of the term containing x^3y^2 and shows the number of selections possible for three x's and two y's.

Example 1

Expand Binomials

coefficients are in row 7:

1 6 15 20 15 6 1

$_6C_0$ $_6C_1$ $_6C_2$ $_6C_3$ $_6C_4$ $_6C_5$ $_6C_6$

a) Expand $(p + q)^6$.

b) Identify patterns in the expansion of $(p + q)^6$.

$$1p^6q^0 + 6p^5q^1 + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6p^1q^5 + 1p^0q^6$$

$$p^6 + 6p^5q + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6pq^5 + q^6$$

b) Some patterns are as follows:

- There are 6 + 1, or 7, terms in the expansion of $(p + q)^6$.
- The powers of p decrease from 6 to 0 in successive terms of the expansion.
- The powers of q increase from 0 to 6.
- Each term is of degree 6 (the sum of the exponents for p and q is 6 for each term)
- The coefficients are symmetrical, 1 6 15 20 15 6 1, and begin and end with 1.

Homework

Expand the following: $(c + d)^{5}$ $n=5$

$${}_5C_0(c^5)(d^0) + {}_5C_1(c^4)(d^1) + {}_5C_2(c^3)(d^2) + {}_5C_3(c^2)(d^3) + {}_5C_4(c^1)(d^4) + {}_5C_5(c^0)(d^5)$$

$$1(c^5)(1) + 5c^4d + 10c^3d^2 + 10c^2d^3 + 5cd^4 + 1(1)d^5$$

$$\boxed{c^5 + 5c^4d + 10c^3d^2 + 10c^2d^3 + 5cd^4 + d^5}$$

Expand the following: $(p+q)^{10}$

binomial theorem

- used to expand $(x + y)^n$, $n \in N$
- each term has the form ${}_nC_k(x)^{n-k}(y)^k$, where $k + 1$ is the term number

You can use the **binomial theorem** to expand any power of a binomial expression.

$$(x + y)^n = {}_nC_0(x)^n(y)^0 + {}_nC_1(x)^{n-1}(y)^1 + {}_nC_2(x)^{n-2}(y)^2 + \cdots + {}_nC_{n-1}(x)^1(y)^{n-1} + {}_nC_n(x)^0(y)^n$$

In this chapter, all binomial expansions will be written in descending order of the exponent of the first term in the binomial.

The following are some important observations about the expansion of $(x + y)^n$, where x and y represent the terms of the binomial and $n \in N$:

- the expansion contains $n + 1$ terms
- the number of objects, k, selected in the combination ${}_nC_k$ can be taken to match the number of factors of the second variable selected; that is, it is the same as the exponent on the second variable
- the general term, t_{k+1}, has the form

$${}_nC_k(x)^{n-k}(y)^k$$

the same

- the sum of the exponents in any term of the expansion is n

Example 2

Use the Binomial Theorem

a) Use the binomial theorem to expand $(2a - 3b)^4$.
b) What is the third term in the expansion of $(4b - 5)^6$?
c) In the expansion of $\left(a^2 - \frac{1}{a}\right)^5$, which term, in simplified form, contains a? Determine the value of the term.

You can use the **binomial theorem** to expand any power of a binomial expression.

$$(x + y)^n = {}_nC_0(x)^n(y)^0 + {}_nC_1(x)^{n-1}(y)^1 + {}_nC_2(x)^{n-2}(y)^2 + \cdots + {}_nC_{n-1}(x)^1(y)^{n-1} + {}_nC_n(x)^0(y)^n$$

a) $(2a - 3b)^4$

$x = 2a$
$y = -3b$
$n = 4$

$${}_4C_0(2a)^4(-3b)^0 + {}_4C_1(2a)^3(-3b)^1 + {}_4C_2(2a)^2(-3b)^2 + {}_4C_3(2a)^1(-3b)^3 + {}_4C_4(2a)^0(-3b)^4$$

$$(1)(16a^4)(1) + (4)(8a^3)(-3b) + (6)(4a^2)(9b^2) + (4)(2a)(-27b^3) + (1)(1)(81b^4)$$

$$16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4$$

b) What is the third term in $(4b - 5)^6$

$x = 4b$
$y = -5$
$n = 6$

$${}_6C_2(4b)^4(-5)^2$$

$$= (15)(256b^4)(25)$$

$$= 96000b^4$$

c) $\left(a^2 - \frac{1}{a}\right)^5$

$x = a^2$
$b = -\frac{1}{a}$
$n = 5$

$${}_5C_0(a^2)^5\left(\tfrac{-1}{a}\right)^0 + {}_5C_1(a^2)^4\left(\tfrac{-1}{a}\right)^1 + {}_5C_2(a^2)^3\left(\tfrac{-1}{a}\right)^2 + {}_5C_3(a^2)^2\left(\tfrac{-1}{a}\right)^3 + {}_5C_4(a^2)^1\left(\tfrac{-1}{a}\right)^4 + {}_5C_5(a^2)^0\left(\tfrac{-1}{a}\right)^5$$

$$(1)(a^{10})(1) + (5)(a^8)\left(\tfrac{-1}{a}\right) + (10)(a^6)\left(\tfrac{1}{a^2}\right) + (10)(a^4)\left(\tfrac{-1}{a^3}\right) + (5)(a^2)\left(\tfrac{1}{a^4}\right) + (1)(1)\left(\tfrac{-1}{a^5}\right)$$

$$a^{10} - \frac{5a^8}{a} + \frac{10a^6}{a^2} - \frac{10a^4}{a^3} + \frac{5a^2}{a^4} - \frac{1}{a^5}$$

4th term

$$a^{10} - 5a^7 + 10a^4 - 10a + \frac{5}{a^2} - \frac{1}{a^5}$$

Key Ideas

- Pascal's triangle has many patterns. For example, each row begins and ends with 1. Each number in the interior of any row is the sum of the two numbers to its left and right in the row above.
- You can use Pascal's triangle or combinations to determine the coefficients in the expansion of $(x + y)^n$, where n is a natural number.
- You can use the binomial theorem to expand any binomial of the form $(x + y)^n$, $n \in \mathrm{N}$.
- You can determine any term in the expansion of $(x + y)^n$ using patterns without having to perform the entire expansion. The general term, t_{k+1}, has the form ${}_nC_k(x)^{n-k}(y)^k$.

Pascals Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

$_4C_0$ $_4C_1$ $_4C_2$ $_4C_3$ $_4C_4$

1 5 10 10 5 1

$_5C_0$ $_5C_1$ $_5C_2$ $_5C_3$ $_5C_4$ $_5C_5$

1 6 15 20 15 6 1

$_6C_0$ $_6C_1$ $_6C_2$ $_6C_3$ $_6C_4$ $_6C_5$ $_6C_6$

Homework

#1-7, 10, 11, 17

Answers to Homework

11.3 The Binomial Theorem, pages 542 to 545

1. a) 1 4 6 4 1 **b)** 1 8 28 56 70 56 28 8 1
c) 1 11 55 165 330 462 462 330 165 55 11 1

2. a) ${}_2C_0\ {}_2C_1\ {}_2C_2$ **b)** ${}_4C_0\ {}_4C_1\ {}_4C_2\ {}_4C_3\ {}_4C_4$
c) ${}_7C_0\ {}_7C_1\ {}_7C_2\ {}_7C_3\ {}_7C_4\ {}_7C_5\ {}_7C_6\ {}_7C_7$

3. a) $\frac{3!}{2!1!}$ **b)** $\frac{6!}{3!3!}$ **c)** $\frac{1!}{0!1!}$

4. a) 5 **b)** 8 **c)** $q + 1$

5. a) $1x^2 + 2xy + 1y^2$ **b)** $1a^3 + 3a^2 + 3a + 1$
c) $1 - 4p + 6p^2 - 4p^3 + 1p^4$

6. a) $1a^3 + 9a^2b + 27ab^2 + 27b^3$
b) $243a^5 - 810a^4b + 1080a^3b^2 - 720a^2b^2 + 240ab^4 - 32b^5$
c) $16x^4 - 160x^3 + 600x^2 - 1000x + 625$

7. a) $126a^4b^5$ **b)** $-540x^3y^3$ **c)** $192\ 192t^6$
d) $96x^2y^2$ **e)** $3072w^2$

8. All outside numbers of Pascal's triangle are 1's; the middle values are determined by adding the two numbers to the left and right in the row above.

9. a) 1, 2, 4, 8, 16
b) 2^8 or 256
c) 2^{n-1}, where n is the row number

10. a) The sum of the numbers on the handle equals the number on the blade of each hockey stick.
b) No; the hockey stick handle must begin with 1 from the outside of the triangle and move diagonally down the triangle with each value being in a different row. The number of the blade must be diagonally below the last number on the handle of the hockey stick.

11. a) 13 **b)** $220x^9y^3$ **c)** $r = 6$, ${}_{12}C_6 = 924$

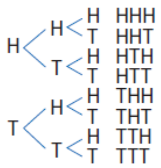
12. a) $(x + y)^4$ **b)** $(1 - y)^5$

13. a) No. While $11^0 = 1$, $11^1 = 11$, $11^2 = 121$, $11^3 = 1331$, and $11^4 = 14\ 641$, this pattern only works for the first five rows of Pascal's triangle.
b) m represents the row number minus 1, $m \le 4$.

14. a) $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$, $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$; the signs for the second and fourth terms are negative in the expansion of $(x - y)^3$
b) $(x + y)^3 + (x - y)^3$
$= x^3 + 3x^2y + 3xy^2 + y^3 + x^3 - 3x^2y + 3xy^2 - y^3$
$= 2x^3 + 6xy^2$
$= 2x(x^2 + 3y^2)$
c) $2y(3x^2 + y^2)$; the expansion of $(x + y)^3 - (x - y)^3$ has coefficients for x^2 and y^2 that are reversed from the expansion of $(x + y)^3 + (x - y)^3$, as well as the common factors $2x$ and $2y$ being reversed.

15. a) Case 1: no one attends, case 2: one person attends case 3: two people attend, case 4: three people attend, case 5: four people attend, case 6: all five people attend
b) 32 or 2^5
c) The answer is the sum of the terms of the sixth row of Pascal's triangle.

16. a)



b) HHH + HHT + HTH + HTT + THH + THT + TTH + TTT
$= H^3 + 3H^2T + 3HT^2 + T^3$
c) H^3 represents the first term of the expansion of $(H + T)^3$ and $3H^2T$ represents the second term of the expansion of $(H + T)^3$.

17. a) $\frac{a^3}{b^3} + 6\left(\frac{a^2}{b^2}\right) + 12\left(\frac{a}{b}\right) + 8$ or $\frac{a^3}{b^3} + \frac{6a^2}{b^2} + \frac{12a}{b} + 8$
b) $\frac{a^4}{b^4} - 4\left(\frac{a^4}{b^3}\right) + 6\left(\frac{a^4}{b^2}\right) - 4\left(\frac{a^4}{b}\right) + a^4$
$= a^4\left(\frac{1}{b^4} - \frac{4}{b^3} + \frac{6}{b^2} - \frac{4}{b} + 1\right)$
c) $1 - 3x + \frac{15}{4}x^2 - \frac{5}{2}x^3 + \frac{15}{16}x^4 - \frac{3}{16}x^5 + \frac{1}{64}x^6$
d) $16x^8 - 32x^5 + 24x^2 - 8x^{-1} + x^{-4}$

18. a) $5670a^4b^{12}$ **b)** the fourth term; it is $-120x^{11}$

19. a) 126 720 **b)** the fifth term; its value is 495