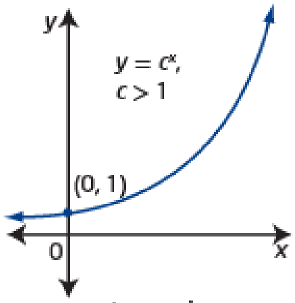
Exponential Functions

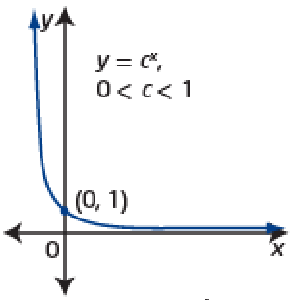
The graph of an **exponential function**, such as $y = c^x$, is increasing for $c > 1$, decreasing for $0 < c < 1$, and neither increasing nor decreasing for $c = 1$. From the graph, you can determine characteristics such as domain and range, any intercepts, and any asymptotes.

exponential function

- a function of the form $y = c^x$, where c is a constant ($c > 0$) and x is a variable



Increasing



Decreasing

Why is the definition of an exponential function restricted to positive values of c?

Did You Know?

Any letter can be used to represent the base in an exponential function. Some other common forms are $y = a^x$ and $y = b^x$. In this chapter, you will use the letter c. This is to avoid any confusion with the transformation parameters, a, b, h, and k, that you will apply in Section 7.2.

Example 3

Application of an Exponential Function

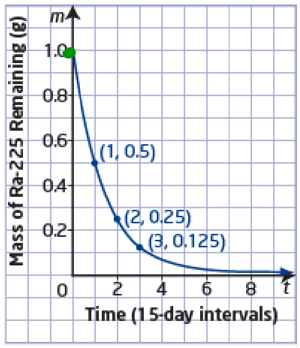
A radioactive sample of radium (Ra-225) has a **half-life** of 15 days. The mass, m, in grams, of Ra-225 remaining over time, t, in 15-day intervals, can be modelled using the exponential graph shown.

a) What is the initial mass of Ra-225 in the sample? What value does the mass of Ra-225 remaining approach as time passes?

b) What are the domain and range of this function?

c) Write the exponential decay model that relates the mass of Ra-225 remaining to time, in 15-day intervals.

d) Estimate how many days it would take for Ra-225 to decay to $\frac{1}{30}$ of its original mass.



a) Initial Amount = 1g
As time passes the radium approaches a mass of 0g.

b) D: $\{x \mid x \geq 0, x \in R\}$ or $[0, \infty)$

R: $\{y \mid 0 < y \leq 1, y \in R\}$ or $(0, 1]$

c) $m = (\text{Initial Amount})(\text{Base})^{\frac{t}{\text{time it takes to..}}}$ $= 15$

$m = (1)\left(\frac{1}{2}\right)^{\frac{t}{15}}$

Base = $\frac{1}{2}$ (Half life)

d) $\dfrac{\frac{1}{30}}{1} = \dfrac{\cancel{(1)}\left(\frac{1}{2}\right)^{\frac{t}{15}}}{\cancel{1}}$ (Divide both sides by Initial Amount)

$\frac{1}{30} = \left(\frac{1}{2}\right)^{\frac{t}{15}}$ (Get a common base)

$\frac{\log\left(\frac{1}{30}\right)}{\log\left(\frac{1}{2}\right)} = \underline{\underline{4.91}}$

$\left(\cancel{\tfrac{1}{2}}\right)^{4.91} = \left(\cancel{\tfrac{1}{2}}\right)^{\frac{t}{15}}$ (Drop the base)

$15 \cdot 4.91 = \frac{t}{15} \cdot 15$ (Solve for the unknown)

$\boxed{73.6 = t}$

73.6 days to reach $\frac{1}{30}$ of its initial amount

So, given that the original value is 1.5, Initial Amount = 1.5

- if we know that the value **doubles** in 5 years, the equation is: $V = 1.5(2)^{\frac{x}{5}}$.

  Base = 2 exp = $\frac{t}{5}$

- if we know that the value **doubles** in 11 years, the equation is: $V = 1.5(2)^{\frac{x}{11}}$.

  Base = 2 exp = $\frac{t}{11}$

- if we know that the value **triples** in 7 years, the equation is: $V = 1.5(3)^{\frac{x}{7}}$.

  Base = 3 exp = $\frac{t}{7}$

Example 2 Initial Amount = 13.5 Base = 2 exp = $\frac{t}{7}$

Anita purchased a book for \$13.50 in 1990. If the value of the book doubled every 7 years, how much would it be worth in 4 years, 11 years, 50 years?

Solution: $V = (\text{Initial Amount})(\text{Base})^{\text{exp}}$

Since it states the value is doubled we can write the equation as: $V = 13.50 \cdot 2^{\frac{x}{7}}$.

So: after 4 years $V = 13.50 \cdot 2^{\frac{4}{7}} = \$20.06$

after 11 years $V = 13.50 \cdot 2^{\frac{11}{7}} = \$40.12$

after 50 years $V = 13.50 \cdot 2^{\frac{50}{7}} = \$1907.86$

a) $V = 13.50(2)^{\left(\frac{4}{7}\right)} = \$20.06$

b) $V = 13.50(2)^{\left(\frac{11}{7}\right)} = \$40.12$

c) $V = 13.50(2)^{\left(\frac{50}{7}\right)} = \$1907.86$

Example 3

Initial Amount = 2300 Base = 3 exp = $\frac{t}{4}$

A culture is found to have 2300 bacteria. The number of bacteria triples in 4 h. Find the amount of bacteria at the end of one day. $(t = 24)$

Solution $A = (\text{Initial Amount})(\text{Base})^{\text{exp}}$ $A = 2300(3)^{\frac{t}{4}}$

The equation for this will be: $A = 2300 \cdot 3^{\frac{x}{4}}$, where x is the # of hours. We use a base of 3 since we are given the tripling time.

So: In 24 hours: $A = 2300 \cdot 3^{\frac{24}{4}} = 1676700$ bacteria.

The three examples above are each exponential functions that exhibit **exponential growth.** We now look at some applications of exponential functions as they relate to **exponential decay.**

Ex: How long until 1000000 bacteria are present? (Find t if A = 1000000)

$A = 2300(3)^{\frac{t}{4}}$

$\frac{1000000}{2300} = \frac{\cancel{2300}(3)^{\frac{t}{4}}}{\cancel{2300}}$ (Divide by I.A.)

$434.78 = 3^{t/4}$ (Get a common base) $\frac{\log(434.78)}{\log(3)}$

$\cancel{3}^{5.53} = \cancel{3}^{t/4}$ (Drop the Base)

$4 \cdot 5.53 = \frac{t}{\cancel{4}} \cdot \cancel{4}$ (Solve for unknown)

$22.12 = t$

hours

Homework

Omit #29 + 36

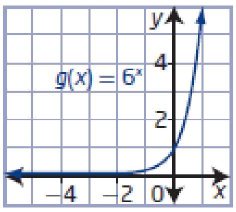
7.1 Characteristics of Exponential Functions, pages 342 to 345

1. a) No, the variable is not the exponent.
b) Yes, the base is greater than 0 and the variable is the exponent.
c) No, the variable is not the exponent.
d) Yes, the base is greater than 0 and the variable is the exponent.

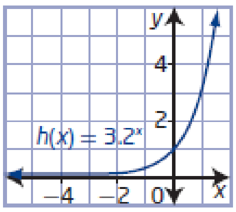
2. a) $f(x) = 4^x$ **b)** $g(x) = \left(\frac{1}{4}\right)^x$
c) $x = 0$, which is the y-intercept

3. a) B **b)** C **c)** A

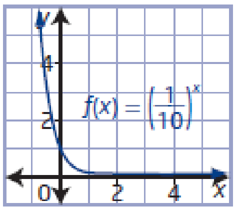
4. a) $f(x) = 3^x$ **b)** $f(x) = \left(\frac{1}{5}\right)^x$

5. a)


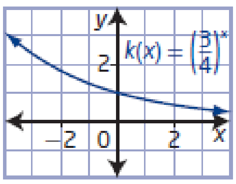
domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y > 0, y \in \mathbb{R}\}$, y-intercept 1, function increasing, horizontal asymptote $y = 0$

b)


domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y > 0, y \in \mathbb{R}\}$, y-intercept 1, function increasing, horizontal asymptote $y = 0$

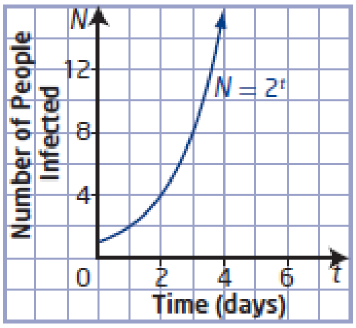
c)


domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y > 0, y \in \mathbb{R}\}$, y-intercept 1, function decreasing, horizontal asymptote $y = 0$

d)


domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y > 0, y \in \mathbb{R}\}$, y-intercept 1, function decreasing, horizontal asymptote $y = 0$

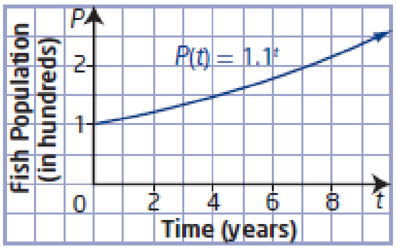
6. a) $c > 1$; number of bacteria increases over time
b) $0 < c < 1$; amount of actinium-225 decreases over time
c) $0 < c < 1$; amount of light decreases with depth
d) $c > 1$; number of insects increases over time

7. a) 

The function $N = 2^t$ is exponential since the base is greater than zero and the variable t is an exponent.

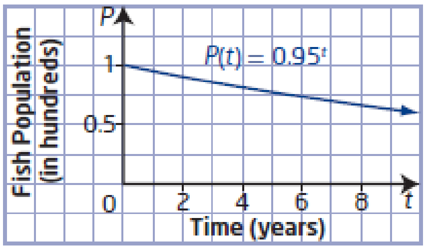
b) **i)** 1 person **ii)** 2 people
iii) 16 people **iv)** 1024 people

8. a) If the population increases by 10% each year, the population becomes 110% of the previous year's population. So, the growth rate is 110% or 1.1 written as a decimal.

b) 

domain $\{t \mid t \geq 0, t \in \mathrm{R}\}$ and range $\{P \mid P \geq 100, P \in \mathrm{R}\}$

c) The base of the exponent would become $100\% - 5\%$ or 95%, written as 0.95 in decimal form.

d) 

domain $\{t \mid t \geq 0, t \in \mathrm{R}\}$ and range $\{P \mid 0 < P \leq 100, P \in \mathrm{R}\}$