Curve Sketching

In this chapter we look at further aspects of curves such as vertical and horizontal asymptotes, concavity, and inflections points. Then we use them, together with intervals of increase and decrease and maximum and minimum values, to develop a procedure for curve sketching.

Questions from Homework

① i) $y = x^3 + 8$

x-intercept $(y=0)$

$0 = x^3 + 8$ (Sum of Cubes)

$0 = (x+2)(x^2 - 2x + 4)$

$x + 2 = 0$

$x = -2$

$(-2, 0)$

$x^2 - 2x + 4 = 0$ $a = 1$ $b = -2$ $c = 4$

$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$

$x = \frac{2 \pm \sqrt{4 - 16}}{2}$

$x = \frac{2 \pm \sqrt{-12}}{2}$

No Solution (2 imaginary roots)

y-intercept $(x=0)$

$y = (0)^3 + 8 = 8$ $(0, 8)$

Vertical Asymptotes

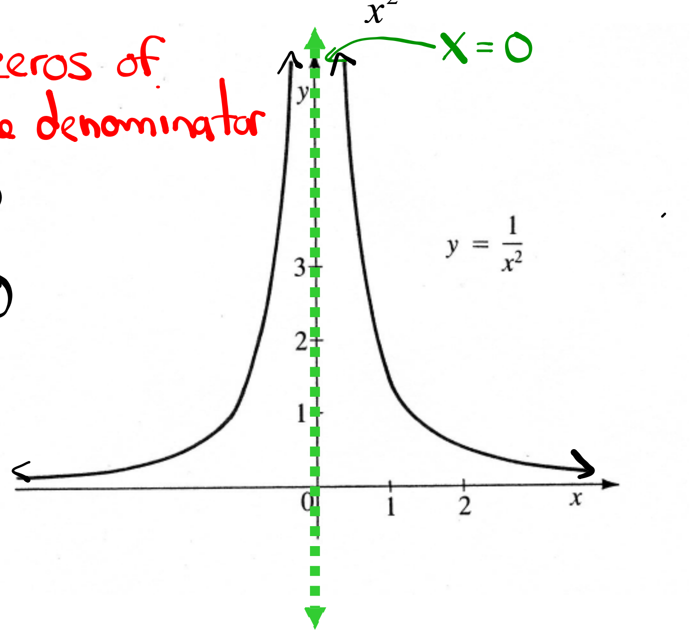
Let us examine the behaviour of the function $f(x)=\frac{1}{x^2}$ for *x* close to 0.

x	$f(x)=\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10000
± 0.001	1000000

VA: → zeros of the denominator

$x^2 = 0$

$x = 0$



The values in the table and the graph show that the closer we take *x* to 0, the larger $\frac{1}{x^2}$ becomes. In fact, it appears that by taking *x* close enough to 0, we can make $f(x)$ as large as we like. We indicate this type of behaviour by writing

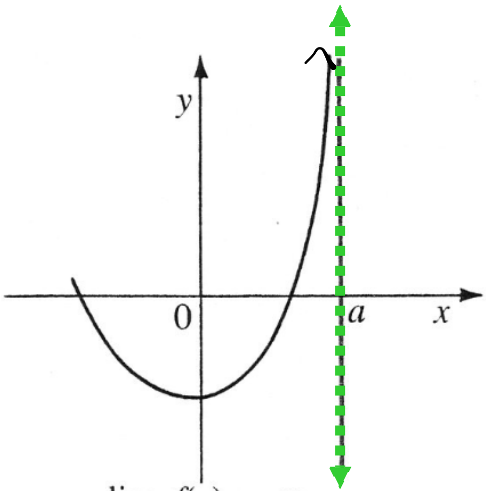
$$\lim_{x\to 0}\frac{1}{x^2}=\infty$$

and we say that the line *x* = *0* is a vertical asymptote of $y=\frac{1}{x^2}$

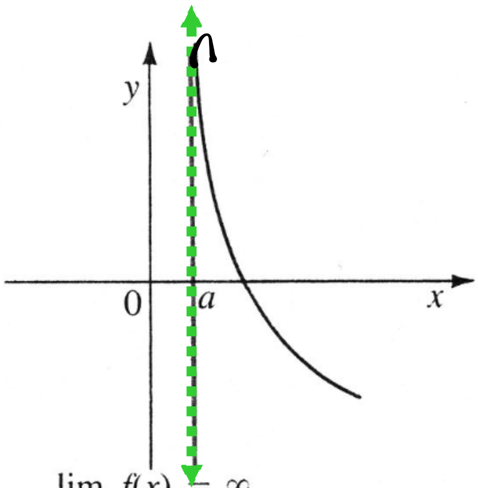
Vertical Asymptote

The line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if either

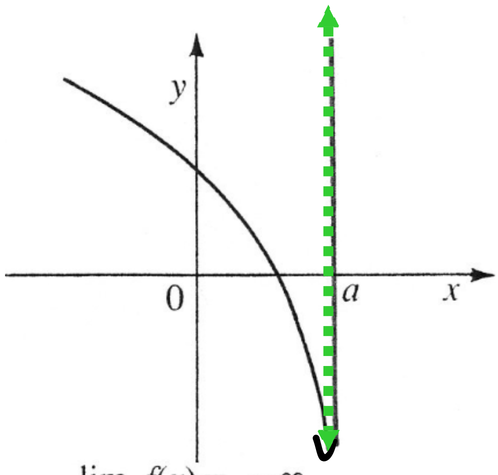
$$\lim_{x \to a^-} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^+} f(x) = \pm\infty$$



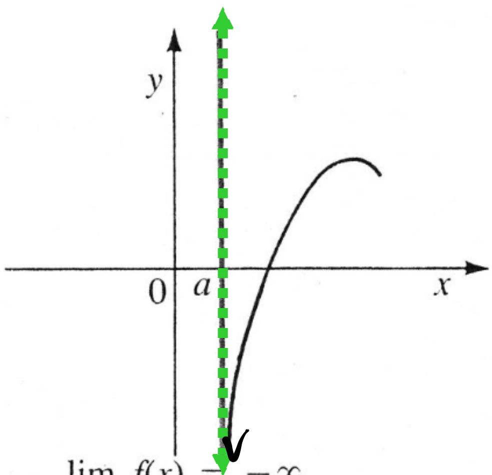
$\lim_{x \to a^-} f(x) = \infty$



$\lim_{x \to a^+} f(x) = \infty$



$\lim_{x \to a^-} f(x) = -\infty$



$\lim_{x \to a^+} f(x) = -\infty$

To find the vertical asymptotes of any rational function, we find the values of *x* where the denominator is zero and compute the limits of the function from the right and left.

Example

a) Find the vertical asymptotes of $y = \frac{x}{x^2 - x - 6}$

b) Sketch the graph near the asymptotes

a) VA are the zeros of the denominator

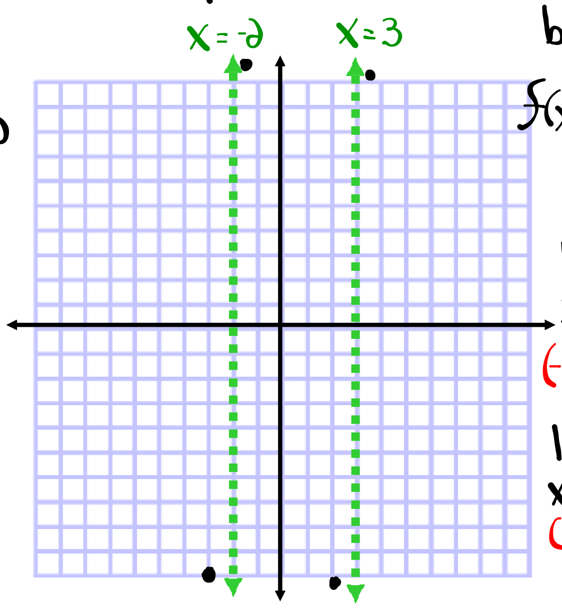
$x^2 - x - 6 = 0$ (Simple trinomial) $2 \times -3 = -6$, $2 + -3 = -1$

$(x+2)(x-3) = 0$

$x+2=0$ | $x-3=0$

$x=-2$ | $x=3$

b)



Use limits to check the behavior near the VA

$f(x) = \frac{x}{(x+2)(x-3)}$ (factored form)

$\lim_{x \to -2^-} f(x) = \frac{(-)}{(-)(-)} = \frac{(-)}{(+)} = -\infty$ (-2.01)

$\lim_{x \to -2^+} f(x) = \frac{(-)}{(+)(-)} = \frac{(-)}{(-)} = +\infty$ (x = -1.99)

$\lim_{x \to 3^-} f(x) = \frac{(+)}{(+)(-)} = \frac{(+)}{(-)} = -\infty$ (x = 2.99)

$\lim_{x \to 3^+} f(x) = \frac{(+)}{(+)(+)} = \frac{(+)}{(+)} = +\infty$ (x = 3.01)

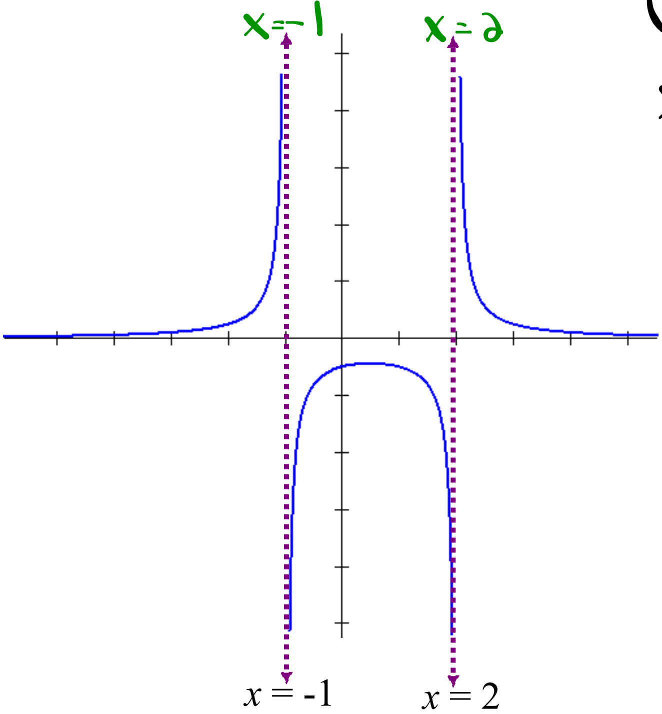
Example: $f(x) = \dfrac{1}{x^2 - x - 2}$

VA: $x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x - 2 = 0$ | $x + 1 = 0$

$x = 2$ | $x = -1$



$\lim_{x \to -1^-} f(x) = +\infty$

$\lim_{x \to -1^+} f(x) = -\infty$

$\lim_{x \to 2^-} f(x) = -\infty$

$\lim_{x \to 2^+} f(x) = +\infty$

Use limits to examine the behaviour of the function near the asymptotes

Homework

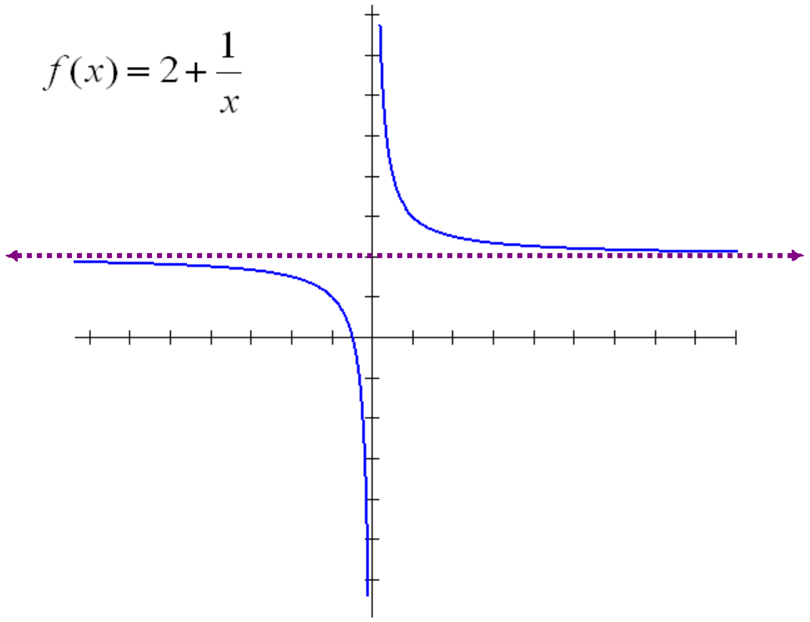
Asymptotes

Horizontal Asymptote

The line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b$$

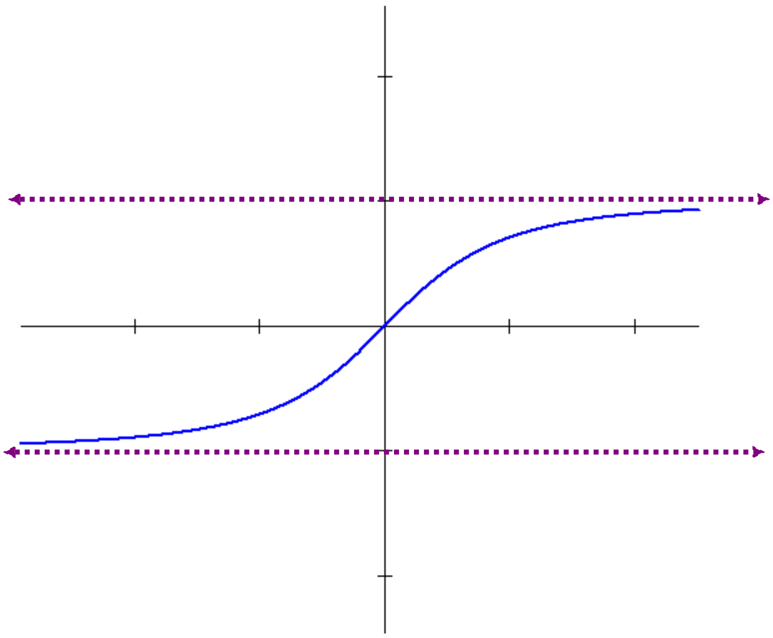
$$f(x) = 2 + \frac{1}{x}$$



Examine the limits of $f(x)$ as x approaches $\pm\infty$

There can be more than one horizontal asymptote.

Examine the function $f(x)=\dfrac{x}{\sqrt{x^2+1}}$



Examine the limits of $f(x)$ as x approaches $\pm\infty$

Sketch the following function:

$$f(x)=\frac{8(x-2)}{x^2} \qquad f'(x)=\frac{-8(x-4)}{x^3} \qquad f''(x)=\frac{16(x-6)}{x^4}$$

Be sure to examine...

- Intercepts
- Asymptotes (*vertical and horizontal)*
- Regions of increase/decrease
- Local extrema
- Regions where concave up/down
- Inflection points