1. [4 points each] Find $\frac{dy}{dx}$ for each of the following. Do **NOT** simplify your answers.

(a) $y = \frac{6 - x^2 + \sqrt[4]{x}}{x^3 + 4}$

$$y' = \frac{(x^3+4)(-2x + \frac{1}{4}x^{-3/4}) - (6 - x^2 + \sqrt[4]{x})(3x^2)}{(x^3+4)^2}$$

(b) $y = e^x \sin(3x^4 + 5)$

$$y' = 12x^3 e^x \cos(3x^4+5) + e^x \sin(3x^4+5)$$

(c) $y = \sec(3x) + \sqrt{\pi x - 5}$

$$y' = 3\sec(3x)\tan(3x) + \frac{1}{2}(\pi x - 5)^{-1/2}(\pi)$$

(b) y

$y' = 2x$

(c) $y =$

$y' = 3se$

(d) $y = 7^x + \cos^6 x$ $= 7^x + (\cos x)^6$

$y' = 7^x \ln 7 - 6\cos^5 x \sin x$

(e) $y = \sin(\tan(2x + 9))$

$y' = 2\cos(\tan(2x+9))\sec^2(2x+9)$

(f) $y = \sin^{-1}(2x) + \ln(x^3 e^x)$

$$y' = \frac{2}{\sqrt{1-4x^2}} + \left(\frac{1}{x^3 e^x}\right)(x^3 e^x + 3x^2 e^x)$$

(g) $y = \left(x^4 + \frac{1}{x^4} + \ln 4\right)^{15}$

$-4x^{-5}$

$y' = 15\left(x^4 + \frac{1}{x^4} + \ln 4\right)^{15}\left(4x^3 - \frac{4}{x^5}\right)$

constant

(h) $y = (x^2+10)^{\cos x}$

$\ln y = \ln (x^2+10)^{\cos x}$

$\ln y = \cos x \ln(x^2+10)$

$\frac{y'}{y} = \cos x\left(\frac{2x}{x^2+10}\right) - \sin x \ln(x^2+10)$

$y' = \left[\frac{2x\cos x}{x^2+10} - \sin x \ln(x^2+10)\right](x^2+10)^{\cos x}$

(i) $3xy + y^4 = x^8 + \sinh x$

$3xy' + 3y + 4y^3y' = 8x^7 + \cosh x$

$y'(3x + 4y^3) = 8x^7 - 3y + \cosh x$

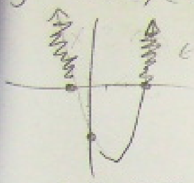
$y' = \frac{8x^7 - 3y + \cosh x}{(3x + 4y^3)}$

2. [4 points] Find the domain of $f(x) = \sqrt{x^2 - 2x - 3}$

$x^2 - 2x - 3 \geq 0$ ← cannot take the square root of a negative

$y = (x-3)(x+1)$

D: $\{x \mid x \leq -1 \text{ and } x \geq 3, x \in \mathbb{R}\}$



3. [3 points] Answer (a) and (b) for the function f defined below.

$$f(x) = \begin{cases} x - 1 & \text{if } x < 1; \\ (x-1)^2 & \text{if } x > 1; \\ 3 & \text{if } x = 1. \end{cases}$$

$x - 1$

x	$f(x)$
1	0
0	-1
-1	-2

$(x-1)^2$

x	$f(x)$
1	0
2	1
3	4

3

x	$f(x)$
1	3

(a) Find $\lim_{x \to 1} f(x)$.

$\lim_{x \to 1^-} f(x)$

$\lim_{x \to 1^-} x - 1 = 0$

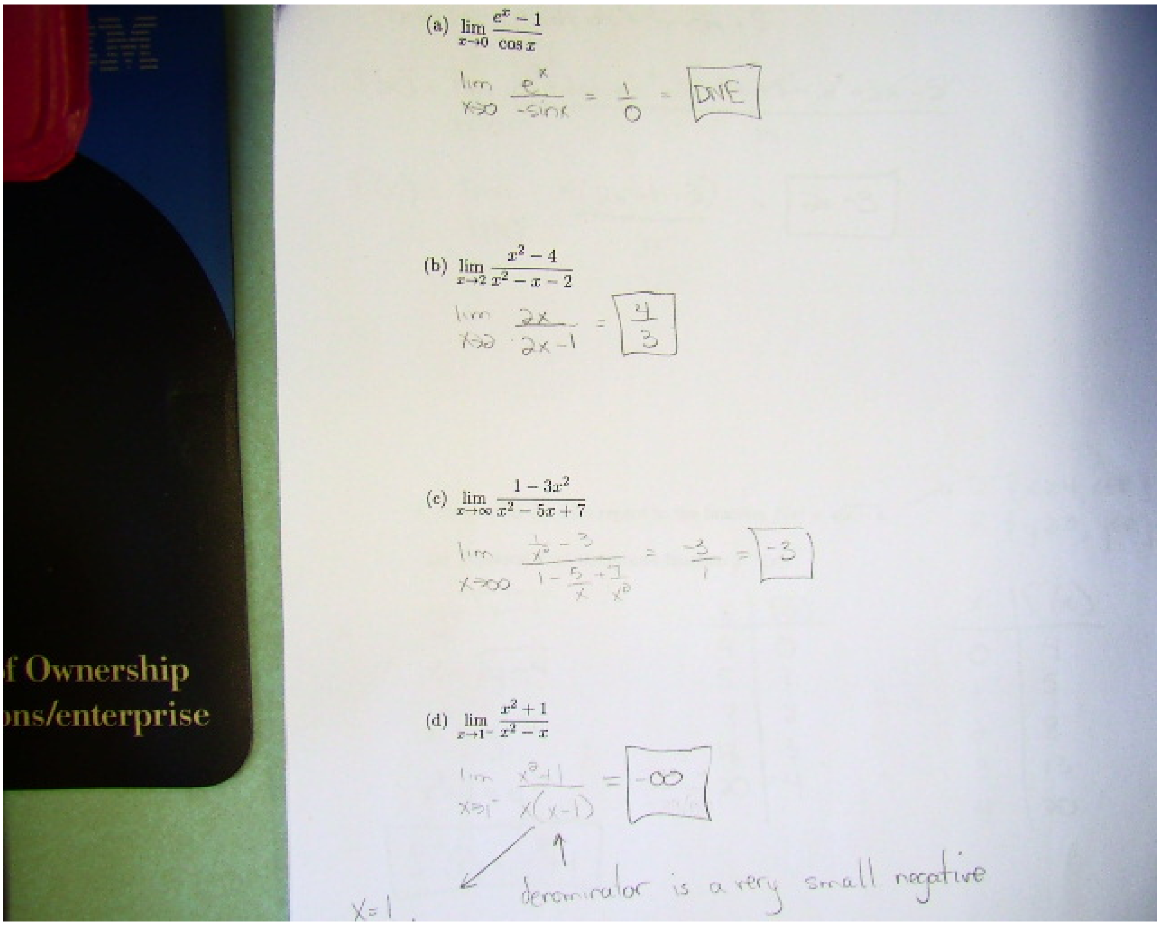
$\lim_{x \to 1^+} f(x)$

$\lim_{x \to 1^+} (x-1)^2 = 0$

$\lim_{x \to 1} f(x) = 0$

(b) Is $f(x)$ continuous at $x = 1$? Justify your answer.

No because $\lim_{x \to 1} f(x) = 0$ and $f(1) = 3$



(a) $\lim_{x\to 0} \frac{e^x - 1}{\cos x}$

$\lim_{x\to 0} \frac{e^x}{-\sin x} = \frac{1}{0} = \boxed{DNE}$

(b) $\lim_{x\to 2} \frac{x^2 - 4}{x^2 - x - 2}$

$\lim_{x\to 2} \frac{2x}{2x-1} = \boxed{\frac{4}{3}}$

(c) $\lim_{x\to \infty} \frac{1 - 3x^2}{x^2 - 5x + 7}$

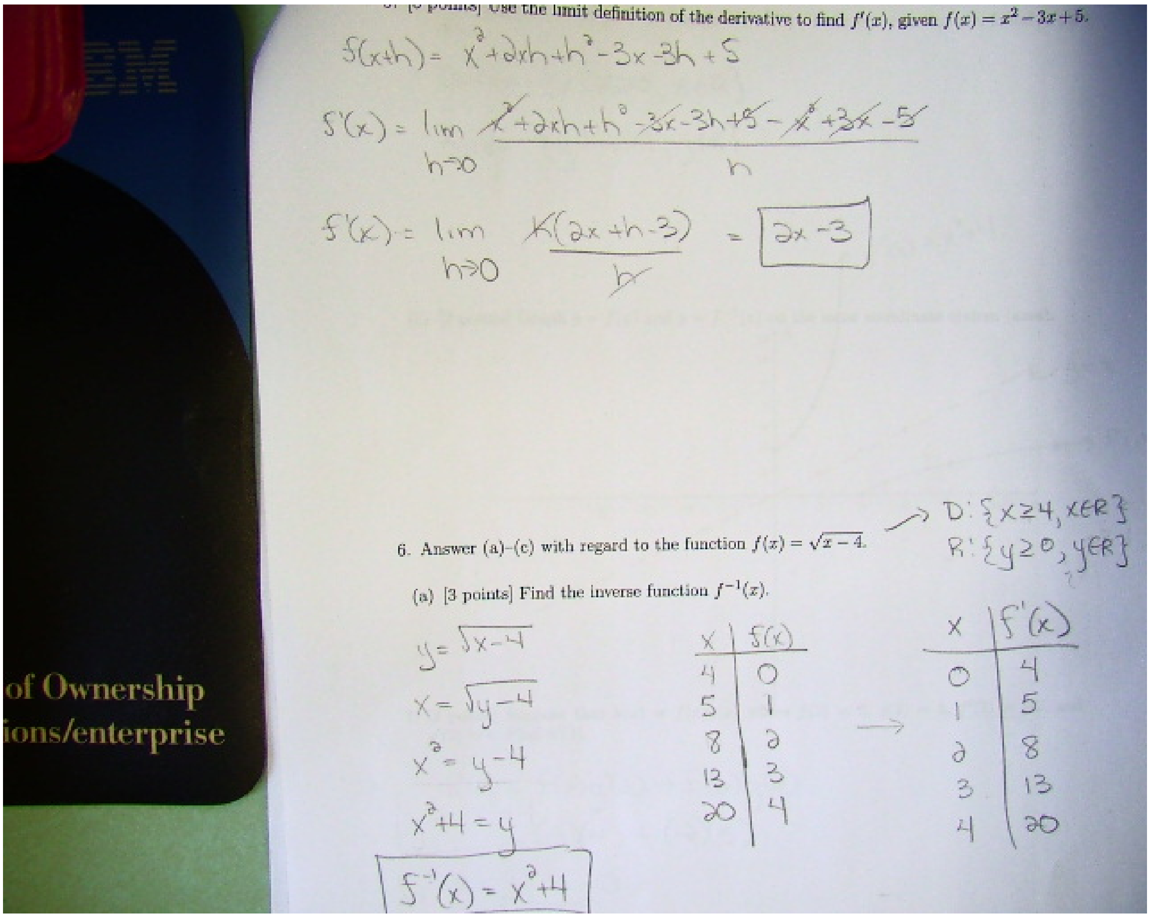
$\lim_{x\to \infty} \frac{\frac{1}{x^2} - 3}{1 - \frac{5}{x} + \frac{7}{x^2}} = \frac{-3}{1} = \boxed{-3}$

(d) $\lim_{x\to 1^-} \frac{x^2 + 1}{x^2 - x}$

$\lim_{x\to 1^-} \frac{x^2+1}{x(x-1)} = \boxed{-\infty}$

$x=1$

denominator is a very small negative



5. [5 points] Use the limit definition of the derivative to find $f'(x)$, given $f(x) = x^2 - 3x + 5$.

$f(x+h) = x^2 + 2xh + h^2 - 3x - 3h + 5$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{-3x} - 3h \cancel{+5} - \cancel{x^2} \cancel{+3x} \cancel{-5}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{h}(2x+h-3)}{\cancel{h}} = \boxed{2x-3}$$

6. Answer (a)–(c) with regard to the function $f(x) = \sqrt{x-4}$. → D: $\{x \geq 4, x \in R\}$ R: $\{y \geq 0, y \in R\}$

(a) [3 points] Find the inverse function $f^{-1}(x)$.

$y = \sqrt{x-4}$

$x = \sqrt{y-4}$

$x^2 = y - 4$

$x^2 + 4 = y$

$\boxed{f^{-1}(x) = x^2 + 4}$

x	f(x)
4	0
5	1
8	2
13	3
20	4

→

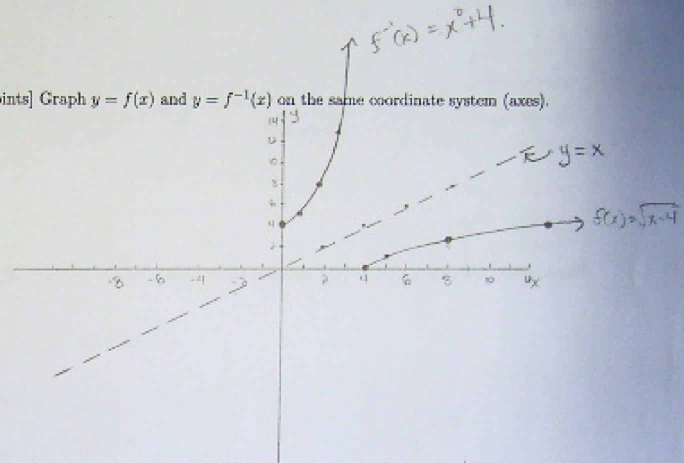
x	f'(x)
0	4
1	5
2	8
3	13
4	20

(b) [2 points] Find the domain and range of $f^{-1}(x)$ (the inverse function you found in part (a)).

Domain $\{x \mid x \geq 0, x \in R\}$

Range $\{y \mid y \geq 4, y \in R\}$

(c) [2 points] Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system (axes).



7. [3 points] Suppose that $h(x) = f(x)g(x)$ where $f(2) = 3$, $g(2) = 5$, $f'(2) = -2$, and $g'(2) = 4$. Find $h'(2)$.

$h'(2) = f(2)g'(2) + f'(2)g(2)$

$= (3)(4) + (-2)(5)$

$= 12 - 10$

$\boxed{= 2}$

8. [5 points] Find the equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.

Point is not on curve

x_1, y_1

$y = x^2 + x$

$y' = \underline{\underline{2x+1}}$

$\uparrow$ m

To Find points of tangency use $y - y_1 = m(x - x_1)$

$y - (-3) = (2x+1)(x-2)$

$y + 3 = 2x^2 - 3x - 2$

$x^2 + x + 3 = 2x^2 - 3x - 2$

$0 = x^2 - 4x - 5$

$0 = (x-5)(x+1)$

$x = 5$ and $x = -1$

Slope @ $x = 5$

$y' = 2(5) + 1 = 11$

Equation

$y + 3 = 11(x - 2)$

$y = 11x - 22 - 3$

$\boxed{y = 11x - 25}$

Slope @ $x = -1$

$y' = 2(-1) + 1 = -1$

Equation

$y + 3 = -1(x - 2)$

$\boxed{y = -x - 1}$

9. [4 points] Find the critical points of the function $f(x) = \ln(2 + \sin x)$ on the interval $[0, 2\pi]$.

$f(x) = \ln(2 + \sin x)$

$f'(x) = \dfrac{1}{2 + \sin x} \cdot \cos x$

$f'(x) = \dfrac{\cos x}{2 + \sin x}$

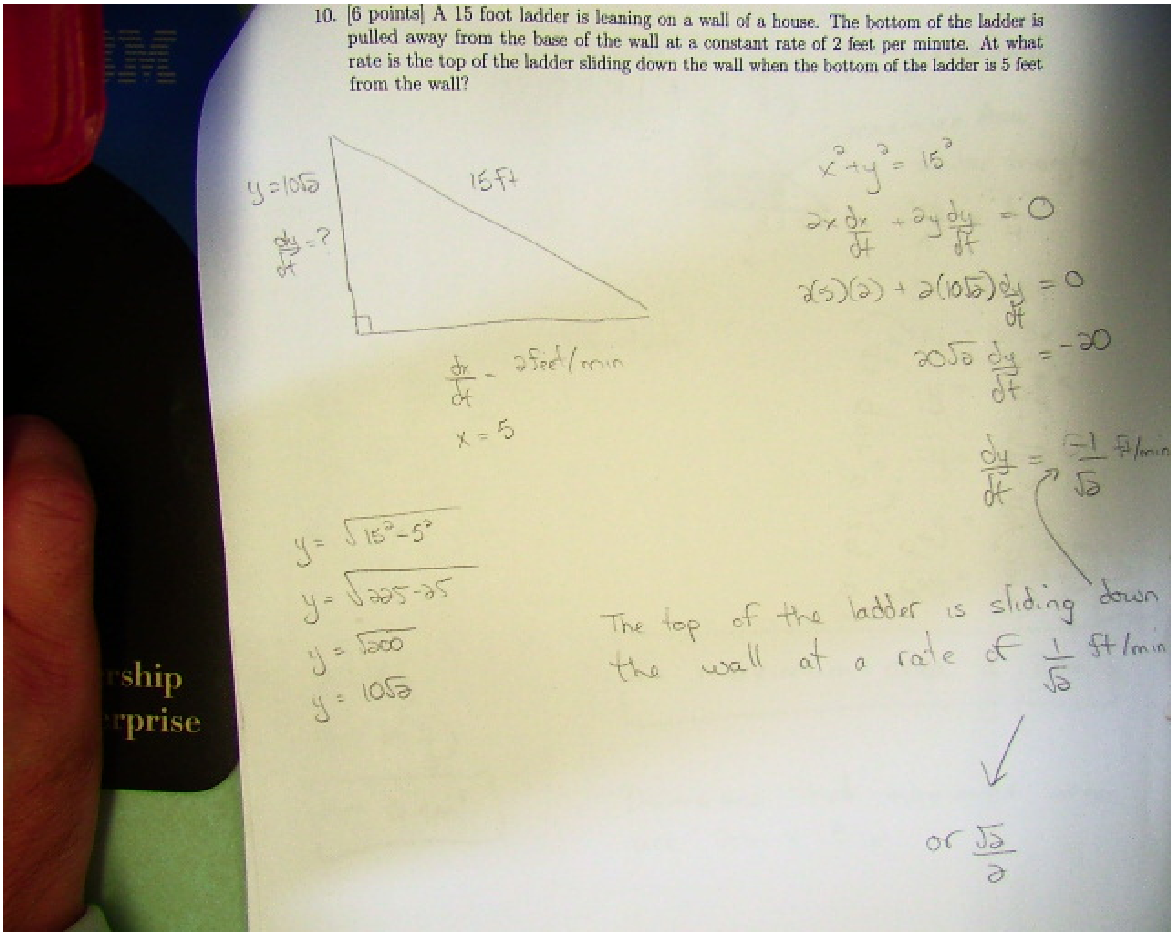
CV: $\cos x = 0$

$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}}$

$2 + \sin x = 0$

$\sin x = -2$

Not possible



10. [6 points] A 15 foot ladder is leaning on a wall of a house. The bottom of the ladder is pulled away from the base of the wall at a constant rate of 2 feet per minute. At what rate is the top of the ladder sliding down the wall when the bottom of the ladder is 5 feet from the wall?

15 ft

$y = 10\sqrt{2}$

$\frac{dy}{dt} = ?$

$\frac{dx}{dt} = 2$ feet/min

$x = 5$

$x^2 + y^2 = 15^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2(5)(2) + 2(10\sqrt{2})\frac{dy}{dt} = 0$

$20\sqrt{2}\,\frac{dy}{dt} = -20$

$\frac{dy}{dt} = \frac{-1}{\sqrt{2}}$ ft/min

$y = \sqrt{15^2 - 5^2}$

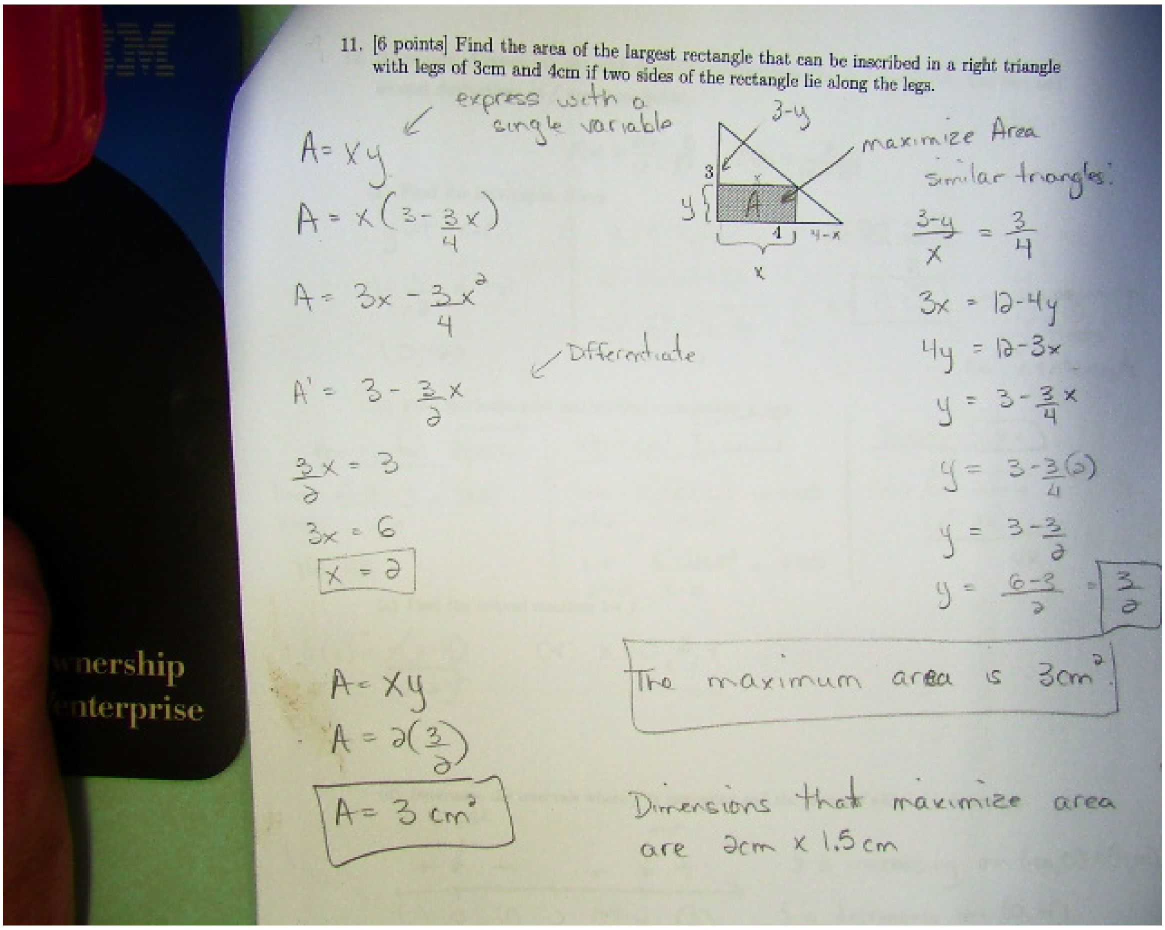
$y = \sqrt{225 - 25}$

$y = \sqrt{200}$

$y = 10\sqrt{2}$

The top of the ladder is sliding down the wall at a rate of $\frac{1}{\sqrt{2}}$ ft/min

or $\frac{\sqrt{2}}{2}$



$A = xy$

$A = x\left(3 - \frac{3}{4}x\right)$

$A = 3x - \frac{3}{4}x^2$

$A' = 3 - \frac{3}{2}x$

$\frac{3}{2}x = 3$

$3x = 6$

$\boxed{x = 2}$

similar triangles:

$\frac{3-y}{x} = \frac{3}{4}$

$3x = 12 - 4y$

$4y = 12 - 3x$

$y = 3 - \frac{3}{4}x$

$y = 3 - \frac{3}{4}(2)$

$y = 3 - \frac{3}{2}$

$y = \frac{6-3}{2} = \boxed{\frac{3}{2}}$

$A = xy$

$A = 2\left(\frac{3}{2}\right)$

$\boxed{A = 3\ cm^2}$

$\boxed{\text{The maximum area is } 3cm^2.}$

Dimensions that maximize area are 2cm x 1.5 cm

12. [13 points] Answer (a)–(h) with regard to the function $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$. The first and second derivatives of f are given below.

$$f'(x) = \frac{x(x-4)}{(x-2)^2} \qquad f''(x) = \frac{8}{(x-2)^3}$$

(a) Find the intercepts, if any.

y int $(x=0)$

$y = \frac{4}{-2} = -2$

$(0,-2)$

x int $(y=0)$

$x^2 - 2x + 4 = 0$

$x = \frac{2 \pm \sqrt{4-16}}{2}$

$x = \frac{2 \pm 2i\sqrt{3}}{2}$

$x = \boxed{1 \pm i\sqrt{3}}$ → Imaginary Roots

No x intercepts

(b) Find the horizontal and vertical asymptotes, if any.

Horizontal $\boxed{\text{None}}$

$\lim_{x\to\infty} \frac{x^2-2x+4}{x-2} = \text{DNE}$

No HA

vertical $\boxed{x=2}$

$\lim_{x\to 2^-} \frac{x^2-2x+4}{x-2} = -\infty$

$\lim_{x\to 2^+} \frac{x^2-2x+4}{x-2} = +\infty$

Slant: $\boxed{y=x}$

$$\begin{array}{r|l} & x \\ x-2 & x^2-2x+4 \\ & -(x^2-2x) \\ \hline & 4R \end{array}$$

(c) Find the critical numbers for f.

$f'(x) = \frac{x(x-4)}{(x-2)^2}$ CV: $x = 0, 2, 4$

(d) Determine the intervals where f is increasing and the intervals where f is decreasing.

max, min

$+ \downarrow - \quad - \downarrow +$

(-1) 0 (1) 2 (3) 4 (5)

$\frac{(-)(-)}{(+)}$ $\frac{(+)(-)}{(+)}$ $\frac{(+)(-)}{(+)}$ $\frac{(+)(+)}{(+)}$

f is increasing on $(-\infty,0) + (4,\infty)$

f is decreasing on $(0,4)$

12

$$f(x) = \frac{x^2-2x+4}{x-2}$$

(e) Find all relative (local) maxima and minima for f.

max $(x=0)$

$f(0) = -2$

$(0,-2)$

min $(x=4)$

$f(4) = \frac{16-8+4}{2} = 6$

$(4,6)$

(f) Determine the intervals where f is concave up and the intervals where f is concave down.

$f''(x) = \frac{8}{(x-2)^3}$

CV's $x=2$

$- \quad \text{ID} \quad +$

$(1) \quad 2 \quad (3)$

$\frac{(+)}{(-)} \quad \frac{(+)}{(+)}$

CU on $(2,\infty)$

CD on $(-\infty,2)$

(g) Find the inflection points, if any.

$f(2) = \frac{4-4+4}{0}$ = undefined

There is no inflection point at $x=2$ because there is a vertical asymptote here

(h) Sketch the graph of $y = f(x)$. Label the intercepts, asymptotes, relative maxima and minima, and inflection points on the graph.