Do I really understand??...

a) Express the following as a single logarithm... $2\log_2 3^2 + \log_2 6 - 3\log_2 3$

b) Evaluate the following... $\log_2 (32)^{\frac{1}{3}} = \frac{1}{3}\boxed{\log_2 32} = \frac{1}{3}(5) = \frac{5}{3}$

c) Express the following as a single logarithm... $\frac{1}{2}[(\log_5 a + 2\log_5 b) - 3\log_5 c]$

d) Express as a single logarithm in simplest form...

$$\frac{3}{4}\left[12(\log_b x^2 - 2\log_b x) + 8\log_b \sqrt{x} - 4\log_b \frac{1}{x^7}\right]$$

$2\log_2 3^2 + \log_2 6 - 3\log_2 3$

$\log_2 3^4 + \log_2 6 - \log_2 3^3$

$\log_2 81 + \log_2 6 - \log_2 27$

$\log_2\left(\frac{81 \cdot 6}{27}\right)$

$\boxed{\log_2 18}$

≈ 4.17

$\frac{1}{2}[(\log_5 a + 2\log_5 b) - 3\log_5 c]$

$\frac{1}{2}[\log_5 a + \log_5 b^2 - \log_5 c^3]$

$\frac{1}{2}\left(\log_5\left(\frac{ab^2}{c^3}\right)\right)$

$\frac{1}{2}\log_5\left(\frac{ab^2}{c^3}\right)$

$\log_5 \sqrt{\frac{ab^2}{c^3}}$

$\log_5 b\sqrt{\frac{a}{c^3}}$

$$\frac{3}{4}\left[12(\log_b x^2 - 2\log_b x) + 8\log_b \sqrt{x} - 4\log_b \frac{1}{x^7}\right]$$

$\frac{3}{4}[12\log_b x^2 - 24\log_b x + 8\log_b x^{1/2} - 4\log_b x^{-7}]$

$9\log_b x^2 - 18\log_b x + 6\log_b x^{1/2} - 3\log_b x^{-7}$

$\cancel{\log_b x^{18}} - \cancel{\log_b x^{18}} + \log_b x^3 - \log_b x^{-21}$

$\log_b\left(\frac{x^3}{x^{-21}}\right)$

$\log_b x^{24}$

$\boxed{24\log_b x}$

Logarithmic and Exponential Equations

Focus on...

- solving a logarithmic equation and verifying the solution
- explaining why a value obtained in solving a logarithmic equation may be extraneous
- solving an exponential equation in which the bases are not powers of one another
- solving a problem that involves exponential growth or decay
- solving a problem that involves the application of exponential equations to loans, mortgages, and investments
- solving a problem by modelling a situation with an exponential or logarithmic equation

General Properties of Logarithms:

If $c > 0$ and $c \neq 1$, then...

(i) $\log_c 1 = 0$

(ii) $\log_c c^x = x$

(iii) $c^{\log_c x} = x$

Did You Know?

The input value for a logarithm is called an argument. For example, in the expression $\log_6 1$, the argument is 1.

Example 1

Solve Logarithmic Equations

Solve.

a) $\log_6 (2x - 1) = \log_6 11$ **b)** $\log (8x + 4) = 1 + \log (x + 1)$

c) $\log_2 (x + 3)^2 = 4$

a) $\cancel{\log_6}(2x-1) = \cancel{\log_6} 11$

$2x - 1 = 11$

$\frac{2x}{2} = \frac{12}{2}$

$x = 6$

is a solution

b) $\log(8x+4) = 1 + \log(x+1)$

$\log(8x+4) - \log(x+1) = 1$

$\log\left(\frac{8x+4}{x+1}\right) = 1$ (log)

$10^1 = \frac{8x+4}{x+1}$ (exp)

$\frac{10}{1} = \frac{8x+4}{x+1}$

$10(x+1) = 8x+4$

$10x + 10 = 8x + 4$

$10x - 8x = 4 - 10$

$\frac{2x}{2} = \frac{-6}{2}$

$x = -3$ extraneous

is not a solution

* c) $\log_2 (x+3)^2 = 4$ (log)

$2^4 = (x+3)^2$ (exp)

$16 = (x+3)(x+3)$

$16 = x^2 + 3x + 3x + 9$

$16 = x^2 + 6x + 9$ bring to one side + Factor

$0 = x^2 + 6x - 7$ $-1 \times 7 = -7$ $-1 + 7 = 6$

$0 = (x-1)(x+7)$

$x - 1 = 0$ | $x + 7 = 0$

$x = 1$ | $x = -7$

is a solution | is also a solution

Example 2

Solve Exponential Equations Using Logarithms

Solve. Round your answers to two decimal places.

a) $4^x = 605$

b) $8(3^{2x}) = 568$

c) $4^{2x-1} = 3^{x+2}$

a) $4^x = 605$

$\log 4^x = \log 605$

$\dfrac{x\cancel{\log 4}}{\cancel{\log 4}} = \dfrac{\log 605}{\log 4}$

$\boxed{x = 4.62}$

b) $\dfrac{\cancel{8}(3^{2x})}{\cancel{8}} = \dfrac{568}{8}$

$3^{2x} = 71$

$\log 3^{2x} = \log 71$

$\dfrac{\cancel{2}x\cancel{\log 3}}{\cancel{2\log 3}} = \dfrac{\log 71}{2\log 3}$

$\boxed{x = 1.94}$

c) $4^{2x-1} = 3^{x+2}$

$\log 4^{2x-1} = \log 3^{x+2}$

$(2x-1)\log 4 = (x+2)\log 3$

$2x\log 4 - \log 4 = x\log 3 + 2\log 3$

Factor

$2x\log 4 - x\log 3 = 2\log 3 + \log 4$

$\dfrac{x\cancel{(2\log 4 - \log 3)}}{\cancel{(2\log 4 - \log 3)}} = \dfrac{(2\log 3 + \log 4)}{(2\log 4 - \log 3)}$

$\boxed{x = 2.14}$

Questions from Homework

(5) a) $2\log_3 x = \log_3 32 + \log_3 2$

$\cancel{\log_3} x^2 = \cancel{\log_3} 64$

$x^2 = 64$

$x = \pm 8$

$\boxed{x = 8}$

c) $\log_2 x - \log_2 3 = 5$

$\log_2\left(\frac{x}{3}\right) = 5$ (log)

Base ans exp

$2^5 = \frac{x}{3}$

$3 \cdot 32 = \frac{x}{3} \cdot 3$

$\boxed{96 = x}$

(7) d) $4(7^{x+2}) = 9^{2x-3}$

$\log 4(7^{x+2}) = \log 9^{2x-3}$

$\log 4 + \log 7^{x+2} = \log 9^{2x-3}$

$\log 4 + (x+2)\log 7 = (2x-3)\log 9$

$\log 4 + x\log 7 + 2\log 7 = 2x\log 9 - 3\log 9$

$x\log 7 - 2x\log 9 = -3\log 9 - \log 4 - 2\log 7$

Factor out an x

$\frac{x(\cancel{\log 7 - 2\log 9})}{(\cancel{\log 7 - 2\log 9})} = \frac{-(3\log 9 + \log 4 + 2\log 7)}{(\log 7 - 2\log 9)}$

$x = \frac{-(3\log 9 + \log 4 + 2\log 7)}{(\log 7 - 2\log 9)}$

$x = \frac{-5.155}{-1.063} = 4.85$

Example 4

Solve a Problem Involving Exponential Growth and Decay

When an animal dies, the amount of radioactive carbon-14 (C-14) in its bones decreases. Archaeologists use this fact to determine the age of a fossil based on the amount of C-14 remaining.
The half-life of C-14 is 5730 years.

Buffalo skull display, Head-Smashed-In buffalo Jump Visitor Centre, near Fort McLeod, Alberta

Head-Smashed-In Buffalo Jump in southwestern Alberta is recognized as the best example of a buffalo jump in North America. The oldest bones unearthed at the site had 49.5% of the C-14 left. How old were the bones when they were found?

Given:

Base = 0.5

$\exp = \frac{t}{5730}$

I.A. = A_0

F.A. = $0.495A_0$

$$y = A_0(0.5)^{\frac{t}{5730}}$$

$$\frac{0.495\cancel{A_0}}{\cancel{A_0}} = \frac{\cancel{A_0}(0.5)^{t/5730}}{\cancel{A_0}}$$

$$0.495 = (0.5)^{t/5730}$$

$$\log 0.495 = \log 0.5^{t/5730}$$

$$5730 \log 0.495 = \frac{t}{\cancel{5730}} \log 0.5 \cdot \cancel{5730}$$

$$\frac{5730 \log 0.495}{\log 0.5} = \frac{t \log 0.5}{\log 0.5}$$

$$\boxed{5813 \text{ yrs} = t}$$

Solution

Carbon-14 decays by one half for each 5730-year interval. The mass, m, remaining at time t can be found using the relationship $m(t) = m_0\left(\frac{1}{2}\right)^{\frac{t}{5730}}$, where m_0 is the original mass.

Since 49.5% of the C-14 remains after t years, substitute $0.495m_0$ for $m(t)$ in the formula $m(t) = m_0\left(\frac{1}{2}\right)^{\frac{t}{5730}}$.

$$\begin{aligned}
0.495m_0 &= m_0\left(\frac{1}{2}\right)^{\frac{t}{5730}} \\
0.495 &= 0.5^{\frac{t}{5730}} \\
\log 0.495 &= \log 0.5^{\frac{t}{5730}} \\
\log 0.495 &= \frac{t}{5730} \log 0.5 \\
\frac{5730 \log 0.495}{\log 0.5} &= t \\
5813 &\approx t
\end{aligned}$$

Instead of taking the common logarithm of both sides, you could have converted from exponential form to logarithmic form. Try this. Which approach do you prefer? Why?

The oldest buffalo bones found at Head-Smashed-In Buffalo Jump date to about 5813 years ago. The site has been used for at least 6000 years.

2. Cesium-137 is an exceptionally dangerous radioactive isotope with a half-life of 30 years. If you have been given a sample of 1600 mg…

a) Write an equation which expresses the mass of Cesium-137 (in mg), as an exponential function of the elapsed time, t (in years).

Given:
I.A. = 1600mg
Base = $\frac{1}{2}$ or 0.5
exp = $\frac{x}{30}$

$$y = 1600\left(\tfrac{1}{2}\right)^{x/30}$$

b) How long will it take for your sample of Cesium-137 to decay to 100 mg?

$$y = 1600(0.5)^{x/30}$$
$$\frac{100}{1600} = \frac{\cancel{1600}(0.5)^{x/30}}{\cancel{1600}}$$
$$0.0625 = (0.5)^{x/30}$$
$$\log 0.0625 = \log(0.5)$$
$$30 \cdot \log 0.0625 = \frac{x}{\cancel{30}}\log 0.5 \cdot \cancel{30}$$
$$\frac{30\log 0.0625}{\log 0.5} = \frac{x\log 0.5}{\log 0.5}$$
$$\boxed{120 = x}$$
120 years

Chapter 7
$$0.0625 = (0.5)^{x/30}$$
$$\cancel{(0.5)}^{4} = \cancel{(0.5)}^{x/30}$$
$$30 \cdot 4 = \frac{x}{30} \cdot 30$$
$$\boxed{120 = x}$$
120 years

c) How much Cesium-137 **(accurate to the nearest hundredth)** will remain after 10 years?

$$y = 1600(0.5)^{x/30}$$
$$y = 1600(0.5)^{10/30}$$
$$y = 1600(0.5)^{1/3} = \boxed{1269.92\text{mg}}$$

Key Ideas

- When solving a logarithmic equation algebraically, start by applying the laws of logarithms to express one side or both sides of the equation as a single logarithm.
- Some useful properties are listed below, where $c, L, R > 0$ and $c \neq 1$.
  - If $\log_c L = \log_c R$, then $L = R$.
  - The equation $\log_c L = R$ can be written with logarithms on both sides of the equation as $\log_c L = \log_c c^R$.
  - The equation $\log_c L = R$ can be written in exponential form as $L = c^R$.
  - The logarithm of zero or a negative number is undefined. To identify whether a root is extraneous, substitute the root into the original equation and check whether all of the logarithms are defined.
- You can solve an exponential equation algebraically by taking logarithms of both sides of the equation. If $L = R$, then $\log_c L = \log_c R$, where $c, L, R > 0$ and $c \neq 1$. Then, apply the power law for logarithms to solve for an unknown.
- You can solve an exponential equation or a logarithmic equation using graphical methods.
- Many real-world situations can be modelled with an exponential or a logarithmic equation. A general model for many problems involving exponential growth or decay is

  $$\text{final quantity} = \text{initial quantity} \times (\text{change factor})^{\text{number of changes}}$$

Key Ideas

- Let P be any real number, and M, N, and c be positive real numbers with $c \neq 1$. Then, the following laws of logarithms are valid.

Name	Law	Description
Product	$\log_c MN = \log_c M + \log_c N$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
Quotient	$\log_c \frac{M}{N} = \log_c M - \log_c N$	The logarithm of a quotient of numbers is the difference of the logarithms of the dividend and divisor.
Power	$\log_c M^P = P \log_c M$	The logarithm of a power of a number is the exponent times the logarithm of the number.

- Many quantities in science are measured using a logarithmic scale. Two commonly used logarithmic scales are the decibel scale and the pH scale.

Homework

Page 412 #1, 2, 4, 5, 7, 8, 11, 15
Chapter 8 Review: #1-14, 18-20, and 22 Page 416

Page 412 (Using method from Ch.7)

⑦ d) $4(7^{x+2}) = 9^{2x-3}$

$\frac{\log 7}{\log 4} = 1.40$

$4 \cdot (4^{1.4})^{x+2} = (4^{1.58})^{2x-3}$

$\frac{\log 9}{\log 4} = 1.58$

$4 \cdot 4^{1.4x+2.8} = 4^{3.16x-4.74}$

$\cancel{4}^{1.4x+3.8} = \cancel{4}^{3.16x-4.74}$

$1.4x + 3.8 = 3.16x - 4.74$

$3.8 + 4.74 = 3.16x - 1.4x$

$\frac{8.54}{1.76} = \frac{1.76x}{1.76}$

$\boxed{4.85 = x}$

Page 413

⑧ c) $2\log_4(x+4) - \log_4(x+12) = 1$

$(x+4)(x+4)$

$x^2+4x+4x+16$

$\log_4(x+4)^2 - \log_4(x+12) = 1$

$\log_4(x^2+8x+16) - \log_4(x+12) = 1$

$\log_4\left(\frac{x^2+8x+16}{x+12}\right) = 1$ (log)

$4^1 = \frac{x^2+8x+16}{x+12}$ (exp)

$(x+12)\ 4 = \frac{x^2+8x+16}{\cancel{x+12}}\ \cancel{(x+12)}$

$4x+48 = x^2+8x+16$

$0 = x^2+4x-32$

$-4 \times 8 = -32$

$-4+8 = 4$

$0 = (x-4)(x+8)$

$x-4=0$ | $x+8=0$

$x=4$ | $x=-8$

is a solution

Review

① a)

$f(x) = 0.2^x$

$f(x) = \left(\frac{1}{5}\right)^x$

Inverse

$\boxed{f^{-1}(x) = \log_{0.2} x}$

x	y
-2	25
-1	5
0	1
1	1/5
2	1/25

x	y
25	-2
5	-1
1	0
1/5	1
1/25	2

b) D: $\{x \mid x > 0, x \in \mathbb{R}\}$

R: $\{y \mid y \in \mathbb{R}\}$

x-int: (1,0)

y-int: None

VA: $x = 0$

④ e) $6^{\log x} = \frac{1}{36}$

$\cancel{6}^{\log x} = \cancel{6}^{-2}$

$\log x = -2$ (log)

$10^{-2} = x$ (exp)

$\frac{1}{100} = x$

⑱ b) $7^{x+1} = 4^{2x-1}$

$\log 7^{x+1} = \log 4^{2x-1}$

$(x+1)\log 7 = (2x-1)\log 4$

$x\log 7 + \log 7 = 2x\log 4 - \log 4$

$\log 7 + \log 4 = 2x\log 4 - x\log 7$

$\frac{\log 7 + \log 4}{2\log 4 - \log 7} = \frac{x\cancel{(2\log 4 - \log 7)}}{\cancel{2\log 4 - \log 7}}$

$\frac{(\log 7 + \log 4)}{(2\log 4 - \log 7)} = x$

$4.03 = x$

⑧ b) $y+7 = \frac{\log_5(6-x)}{4}$

$y = \frac{1}{4}\log_5(-x+6) - 7$

$y = \frac{1}{4}\log_5(-1(x-6)) - 7$

$a = \frac{1}{4}$ → vertical stretch by a factor of $\frac{1}{4}$

$b = -1$ → horizontal reflection in the y axis.

$h = 6$ → horizontally translated 6 units right

$k = -7$ → vertically " 7 " down.

(19) b) $\log_4(x+2) - \log_4(x-4) = \frac{1}{2}$

$$\log_4\left(\frac{x+2}{x-4}\right) = \frac{1}{2} \qquad (\log)$$

$$4^{1/2} = \frac{x+2}{x-4} \qquad (\exp)$$

$$(x-4)\,2 = \frac{x+2}{\cancel{x-4}}\,\cancel{(x-4)}$$

$$2x - 8 = x + 2$$

$$x = 10$$

(19) 2) $\log\sqrt{x^2-21x} = 1$ (log)

$10^1 = \sqrt{x^2-21x}$ (exp)

$10 = \sqrt{x^2-21x}$ (Square both sides)

$100 = x^2-21x$

$0 = x^2-21x-100$

$0 = (x-25)(x+4)$

$-25 \times 4 = -100$

$-25 + 4 = -21$

1x100
2x50
4x25
5x20
10x10

$x-25=0 \mid x+4=0$

$x=25 \mid x=-4$

Both are solution

20 Depreciates by 32%

Base = 68% = 0.68

I.A. = 1200

exp = $\frac{x}{1}$ = x

$y = 1200(0.68)^x$

$\frac{100}{1200} = \frac{\cancel{1200}(0.68)^x}{\cancel{1200}}$

$0.083 = 0.68^x$

$\log 0.083 = \log 0.68^x$

$\frac{\log 0.083}{\log 0.68} = \frac{x \cancel{\log 0.68}}{\cancel{\log 0.68}}$

$\boxed{6.45 \text{ yrs} = x}$

8.4 Logarithmic and Exponential Equations, pages 412 to 415

1. a) 1000 **b)** 14 **c)** 3 **d)** 108
2. a) 1.61 **b)** 10.38 **c)** 4.13 **d)** 0.94
3. No, since $\log_3 (x - 8)$ and $\log_3 (x - 6)$ are not defined when $x = 5$.
4. a) $x = 0$ is extraneous.
b) Both roots are extraneous.
c) $x = -6$ is extraneous.
d) $x = 1$ is extraneous.
5. a) $x = 8$ **b)** $x = 25$ **c)** $x = 96$ **d)** $x = 9$

6. a) Rubina subtracted the contents of the log when she should have divided them. The solution should be

$$\begin{aligned} \log_6 \left(\frac{2x+1}{x-1}\right) &= \log_6 5 \\ 2x + 1 &= 5(x - 1) \\ 1 + 5 &= 5x - 2x \\ 6 &= 3x \\ x &= 2 \end{aligned}$$

b) Ahmed incorrectly concluded that there was no solution. The solution is $x = 0$.
c) Jennifer incorrectly eliminated the log in the third line. The solution, from the third line on, should be

$$\begin{aligned} x(x + 2) &= 2^3 \\ x^2 + 2x - 8 &= 0 \\ (x - 2)(x + 4) &= 0 \end{aligned}$$

So, $x = 2$ or $x = -4$.
Since $x > 0$, the solution is $x = 2$.

7. a) 0.65 **b)** -0.43 **c)** 81.37 **d)** 4.85
8. a) no solution ($x = -3$ not possible)
b) $x = 10$ **c)** $x = 4$ **d)** $x = 2$ **e)** $x = -8, 4$
9. a) about 2.64 pc **b)** about 8.61 light years
10. 64 kg
11. a) 10 000 **b)** 3.5%
c) approximately 20.1 years
12. a) 248 Earth years **b)** 228 million kilometres
13. a) 2 years **b)** 44 days **c)** 20.5 years
14. 30 years
15. approximately 9550 years
16. 8 days
17. 34.0 m
18. $x = 4.5$, $y = 0.5$
19. a) The first line is not true.
b) To go from line 4 to line 5, you are dividing by a negative quantity, so the inequality sign must change direction.
20. a) $x = 100$ **b)** $x = \frac{1}{100}, 100$ **c)** $x = 1, 100$
21. a) $x = 16$ **b)** $x = 9$
22. $x = -5, 2, 4$

$$y = a\log_c(b(x-h)) + k$$

$$(x, y) \longrightarrow \left(\frac{1}{b}x + h, ay + k\right)$$