Develop the definition of a derivative

The concept of **Derivative** is at the core of Calculus and modern mathematics. The definition of the derivative can be approached in two different ways. One is geometrical (as a slope of a curve) and the other one is physical (as a rate of change). Historically there was (and maybe still is) a fight between mathematicians which of the two illustrates the concept of the derivative best and which one is more useful. We will not dwell on this. Our emphasis will be on the use of the derivative as a tool.

Definition. Let $y = f(x)$ be a function. The **derivative of f** is the function whose value at x is the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

If this limit exists for each x in an open interval **I**, then we say that **f is differentiable on I**.

$f'(x)$ = derivative

= y'

= $\frac{dy}{dx}$

= differentiate

* = the slope of the tangent line

$\boxed{f(x) = y = \text{functions height}}$

when you plug x-values into the f(x) formula you are solving for the height

$\boxed{f'(x) = \text{derivative} = \text{slope of the tangent}}$

when you plug x-values into the f'(x) formula you are solving for a slope (m)

Notation:

$$f'(x) \Leftrightarrow \frac{dy}{dx}$$

$$f''(x) \Leftrightarrow \frac{d^2y}{dx^2}$$

Examples:

Use the definition of a derivative to differentiate...

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

① Find $f(x+h)$
② Fill in your formula
③ Solve

$f(x) = 4x^2 - 4$

① $f(x+h) = 4(x+h)^2 - 4$

$= 4(x+h)(x+h) - 4$

$= 4(x^2 + xh + xh + h^2) - 4$

$= 4(x^2 + 2xh + h^2) - 4$

$= 4x^2 + 8xh + 4h^2 - 4$

② $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$f'(x) = \lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 - 4 - (4x^2 - 4)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{4x^2} + 8xh + 4h^2 \cancel{- 4} \cancel{- 4x^2} \cancel{+ 4}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{8xh + 4h^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{4\cancel{h}(2x + h)}{\cancel{h}} = 4(2x + 0) = \boxed{8x}$$

$f'(x) = 8x$

$f(x) = 4x^2 - 4$

Suppose:

$$f(5) = 4(5)^2 - 4 = 4(25) - 4 = 100 - 4 = 96$$

$$f(-1) = 4(-1)^2 - 4 = 4(1) - 4 = 4 - 4 = 0$$

Examples:

Use the definition of a derivative to differentiate... find $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{\boxed{f(x+h)} - \boxed{f(x)}}{h}$$

$$f(x) = \boxed{2x^2 - 6x + 1}$$

$$f(x+h) = 2(x+h)^2 - 6(x+h) + 1$$
$$= 2(x^2 + 2xh + h^2) - 6x - 6h + 1$$
$$= \boxed{2x^2 + 4xh + 2h^2 - 6x - 6h + 1}$$

$$f'(x) = \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 6x - 6h + 1 - (2x^2 - 6x + 1)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{- 6x} - 6h \cancel{+ 1} \cancel{- 2x^2} \cancel{+ 6x} \cancel{- 1}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{4xh + 2h^2 - 6h}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{h}(4x + 2h - 6)}{\cancel{h}} = \boxed{4x - 6}$$

Examples:

Use the definition of a derivative to differentiate...

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

① Find $f(x+h)$
② fill in your formula
③ solve

$$f(x) = \boxed{\frac{x+1}{3x-2}}$$

$$f(x+h) = \frac{(x+h)+1}{3(x+h)-2} = \boxed{\frac{x+h+1}{3x+3h-2}}$$

$$f'(x) = \lim_{h\to 0} \frac{(3x-2)\cancel{(3x+3h-2)}\frac{x+h+1}{\cancel{3x+3h-2}} - \frac{x+1}{\cancel{3x-2}}\cancel{(3x-2)}(3x+3h-2)}{h\,(3x-2)(3x+3h-2)}$$

$$f'(x) = \lim_{h\to 0} \frac{(3x-2)(x+h+1)-(x+1)(3x+3h-2)}{h(3x-2)(3x+3h-2)}$$

$$f'(x) = \lim_{h\to 0} \frac{3x^2+3xh+3x-2x-2h-2-(3x^2+3xh-2x+3x+3h-2)}{h(3x-2)(3x+3h-2)}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{3x^2}+\cancel{3xh}+\cancel{3x}-\cancel{2x}-2h-\cancel{2}-\cancel{3x^2}-\cancel{3xh}+\cancel{2x}-\cancel{3x}-3h+\cancel{2}}{h(3x-2)(3x+3h-2)}$$

$$f'(x) = \lim_{h\to 0} \frac{-5\cancel{h}}{\cancel{h}(3x-2)(3x+3h-2)} = \frac{-5}{(3x-2)(3x-2)} = \frac{-5}{(3x-2)^2}$$

$$\boxed{f'(x) = \frac{-5}{(3x-2)^2}}$$

Find the derivative of each function.

Remember!

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

1. $f(x) = 8x^2 - 10$

   $f'(x) = 16x$

2. $f(x) = 2x^2 + 14x - 7$

   $f'(x) = 4x + 14$

3. $f(x) = x^3$

   $f'(x) = 3x^2$

4. $f(x) = \dfrac{x+4}{2x+3}$

   $f'(x) = \dfrac{-5}{(2x+3)^2}$