Understanding Logarithms

Focus on...

- demonstrating that a logarithmic function is the inverse of an exponential function
- sketching the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- determining the characteristics of the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- explaining the relationship between logarithms and exponents
- expressing a logarithmic function as an exponential function and vice versa
- evaluating logarithms using a variety of methods

Questions from Homework

③ b) $\log_2 32 = 5$ $\qquad \log_3\left(\frac{1}{27}\right) = -3$

$\frac{\log 32}{\log 2} = 5$ $\qquad \frac{\log\left(\frac{1}{27}\right)}{\log 3} = -3$

j) $\log_9 \sqrt{3} = 0.25$

$\frac{\log(\sqrt{3})}{\log 9} = 0.25$ or $\frac{1}{4}$

④ f) $\underset{\text{base}}{3}^{\overset{\text{exp}}{2x-1}} = \underset{\text{ans}}{5}$ (exp form)

$\log_3(5) = 2x - 1$

$\frac{\log_3(5) + 1}{2} = \frac{2x}{2}$

$\frac{\log_3(5) + 1}{2} = x$

$\frac{1}{2}\left(\log_3(5) + 1\right) = x$

g) $\log_{\underset{\text{Base}}{2}}\left(\underbrace{\log_3 x}_{\text{ans}}\right) = \underset{\text{exp}}{4}$ (log form)

$2^4 = \log_3 x$ (Exp. form)

$\underset{\text{exp}}{16} = \log_{\underset{\text{Base}}{3}} \underset{\text{ans}}{x}$ (log form)

$3^{16} = x$ (Exp. form)

$43\,046\,721 = x$

h) $\underset{\text{Base}}{10}^{\overset{\text{exp}}{5^x}} = \underset{\text{ans}}{3}$ (Exp. form)

$\underbrace{\log_{10}(3)}_{\text{ans}} = \underset{\text{Base}}{5}^{\overset{\text{exp}}{x}}$ (Exp form)

$\log_5(\log_{10} 3) = x$

Questions from Homework

Exercise 3

③ e) $\frac{1}{2}\left[\log_5 x + 2\log_5 y - 3\log_5 z\right]$

$\frac{1}{2}\left[\log_5 x + \log_5 y^2 - \log_5 z^3\right]$

$\frac{1}{2}\log_5\left(\frac{xy^2}{z^3}\right)$

$\log_5\left(\frac{xy^2}{z^3}\right)^{1/2}$

$\log_5\sqrt{\frac{xy^2}{z^3}}$

$\log_5 y\sqrt{\frac{x}{z^3}}$

General Properties of Logarithms:

If $c > 0$ and $c \neq 1$, then...

(i) $\log_c 1 = 0$

(ii) $\log_c c^x = x$

(iii) $c^{\log_c x} = x$

Did You Know?

The input value for a logarithm is called an argument. For example, in the expression $\log_6 1$, the argument is 1.

(i) $\log_5 1 = 0$ (ii) $\log_2 2^3 = 3$ (iii) $7^{\log_7 49} = 49$

$5^{\log_5 10} = 10$

Product Law of Logarithms

The logarithm of a product of numbers can be expressed as the sum of the logarithms of the numbers.

$$\boxed{\log_c MN = \log_c M + \log_c N}$$

$\log_2 16 + \log_2 2 = \log_2(16 \cdot 2)$
$= \log_2 32$
$= 5$

Proof

Let $\log_c M = x$ and $\log_c N = y$, where M, N, and c are positive real numbers with $c \neq 1$.

Write the equations in exponential form as $M = c^x$ and $N = c^y$:

$MN = (c^x)(c^y)$

$MN = c^{x+y}$ Apply the product law of powers.

$\log_c MN = x + y$ Write in logarithmic form.

$\log_c MN = \log_c M + \log_c N$ Substitute for x and y.

Quotient Law of Logarithms

The logarithm of a quotient of numbers can be expressed as the difference of the logarithms of the dividend and the divisor.

$$\boxed{\log_c \frac{M}{N} = \log_c M - \log_c N}$$

ex: $\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right)$

$= \log_2 16$

$= 4$

Proof

Let $\log_c M = x$ and $\log_c N = y$, where M, N, and c are positive real numbers with $c \neq 1$.

Write the equations in exponential form as $M = c^x$ and $N = c^y$:

$$\frac{M}{N} = \frac{c^x}{c^y}$$

$$\frac{M}{N} = c^{x-y} \qquad \text{Apply the quotient law of powers.}$$

$$\log_c \frac{M}{N} = x - y \qquad \text{Write in logarithmic form.}$$

$$\log_c \frac{M}{N} = \log_c M - \log_c N \qquad \text{Substitute for } x \text{ and } y.$$

Power Law of Logarithms

The logarithm of a power of a number can be expressed as the exponent times the logarithm of the number.

$$\boxed{\log_c M^P = P \log_c M}$$

How could you prove the quotient law using the product law and the power law?

Proof

Let $\log_c M = x$, where M and c are positive real numbers with $c \neq 1$.

Write the equation in exponential form as $M = c^x$.

Let P be a real number.

$$\begin{aligned}
M &= c^x \\
M^P &= (c^x)^P \\
M^P &= c^{xP} && \text{Simplify the exponents.} \\
\log_c M^P &= xP && \text{Write in logarithmic form.} \\
\log_c M^P &= (\log_c M)P && \text{Substitute for } x. \\
\log_c M^P &= P \log_c M
\end{aligned}$$

The laws of logarithms can be applied to logarithmic functions, expressions, and equations.

ex: $\log_3 \sqrt[4]{27}$

$= \log_3 (27)^{\frac{1}{4}}$

$= \frac{1}{4} \boxed{\log_3 27}$

$= \frac{1}{4} \boxed{\frac{(3)}{1}}$

$= \frac{3}{4}$

Product Law of Logarithms

The logarithm of a product of numbers can be expressed as the sum of the logarithms of the numbers.

$$\boxed{\log_c MN = \log_c M + \log_c N}$$

Quotient Law of Logarithms

The logarithm of a quotient of numbers can be expressed as the difference of the logarithms of the dividend and the divisor.

$$\boxed{\log_c \frac{M}{N} = \log_c M - \log_c N}$$

Power Law of Logarithms

The logarithm of a power of a number can be expressed as the exponen times the logarithm of the number.

$$\boxed{\log_c M^P = P \log_c M}$$

How could you prove the quotient law using the product law and the power law?

Example 1

Use the Laws of Logarithms to Expand Expressions

Write each expression in terms of individual logarithms of x, y, and z.

a) $\log_5 \frac{xy}{z}$

b) $\log_7 \sqrt[3]{x}$

c) $\log_6 \frac{1}{x^2}$

d) $\log \frac{x^3}{y\sqrt{z}}$

a) $\log_5 \frac{xy}{z} = \log_5 x + \log_5 y - \log_5 z$

b) $\log_7 \sqrt[3]{x} = \log_7 x^{\frac{1}{3}} = \frac{1}{3}\log_7 x$

c) $\log_6 \frac{1}{x^2} = \log_6 1 - \log_6 x^{2}$

$= 0 - 2\log_6 x$

$= -2\log_6 x$

d) $\log\left(\frac{x^3}{y\sqrt{z}}\right)$

$= \log\left(\frac{x^3}{y z^{1/2}}\right)$

$= \log x^3 - \log y - \log z^{1/2}$

$= 3\log x - \log y - \frac{1}{2}\log z$

Example 2

Use the Laws of Logarithms to Evaluate Expressions

Use the laws of logarithms to simplify and evaluate each expression.

a) $\log_6 8 + \log_6 9 - \log_6 2$

b) $\log_7 7\sqrt{7}$

c) $2\log_2 12 - \left(\log_2 6 + \frac{1}{3}\log_2 27\right)$

a) $\log_6 8 + \log_6 9 - \log_6 2 = \log_6\left(\frac{8\cdot 9}{2}\right) = \log_6(36) = 2$

b) $\log_7 7\sqrt{7} = \log_7 7(7)^{1/2} = \log_7 7^{3/2} = \frac{3}{2}$

$\hookrightarrow \frac{3}{2}(\log_7 7)$

$\frac{3}{2}(1)$

$\frac{3}{2}$

c) $2\log_2 12 - \left(\log_2 6 + \frac{1}{3}\log_2 27\right)$

$\log_2 12^2 - \left(\log_2 6 + \log_2 27^{1/3}\right)$

$= \log_2 144 - \left(\log_2 6 + \log_2 3\right)$

$= \log_2 144 - \log_2 6 - \log_2 3$

$= \log_2\left(\frac{144}{6\cdot 3}\right)$

$= \log_2 8$

$= 3$

Example 3

Use the Laws of Logarithms to Simplify Expressions

Write each expression as a single logarithm in simplest form. State the restrictions on the variable.

a) $\log_7 x^2 + \log_7 x - \dfrac{5 \log_7 x}{2}$

b) $\log_5 (2x - 2) - \log_5 (x^2 + 2x - 3)$

a) $\log_7 x^2 + \log_7 x - \frac{5}{2}\log_7 x$

$\log_7 x^2 + \log_7 x - \log_7 x^{5/2}$

$\log_7 \left(\frac{x^2 \cdot x}{x^{5/2}}\right) \rightarrow 3 - \frac{5}{2}$

$\frac{6}{2} - \frac{5}{2}$

$\log_7 x^{1/2}$

b) $\log_5 (2x-2) - \log_5 (x^2+2x-3)$

$\log_5 \left(\frac{2x-2}{x^2+2x-3}\right)$

$\log_5 \left(\frac{2\cancel{(x-1)}}{\cancel{(x-1)}(x+3)}\right)$

$\log_5 \left(\frac{2}{x+3}\right)$

For the original expression to be defined, both logarithmic terms must be defined.

$2x - 2 > 0$ $\quad$ $x^2 + 2x - 3 > 0$

$2x > 2$ $\quad$ $(x + 3)(x - 1) > 0$

$x > 1$ and $x < -3$ or $x > 1$

What other methods could you have used to solve this quadratic inequality?

The conditions $x > 1$ and $x < -3$ or $x > 1$ are both satisfied when $x > 1$.

Hence, the variable x needs to be restricted to $x > 1$ for the original expression to be defined and then written as a single logarithm.

Therefore, $\log_5 (2x - 2) - \log_5 (x^2 + 2x - 3) = \log_5 \frac{2}{x + 3}$, $x > 1$.

Key Ideas

- Let P be any real number, and M, N, and c be positive real numbers with $c \neq 1$. Then, the following laws of logarithms are valid.

Name	Law	Description
Product	$\log_c MN = \log_c M + \log_c N$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
Quotient	$\log_c \frac{M}{N} = \log_c M - \log_c N$	The logarithm of a quotient of numbers is the difference of the logarithms of the dividend and divisor.
Power	$\log_c M^P = P \log_c M$	The logarithm of a power of a number is the exponent times the logarithm of the number.

- Many quantities in science are measured using a logarithmic scale. Two commonly used logarithmic scales are the decibel scale and the pH scale.

Homework

Exercise 3

Do I really understand??...

a) Express the following as a single logarithm... $2\log_2 3^2 + \log_2 6 - 3\log_2 3$

b) Evaluate the following... $\log_2 (32)^{\frac{1}{3}}$

c) Express the following as a single logarithm... $\frac{1}{2}\left[(\log_5 a + 2\log_5 b) - 3\log_5 c\right]$

d) Express as a single logarithm in simplest form...

$$\frac{3}{4}\left[12\left(\log_b x^2 - 2\log_b x\right) + 8\log_b \sqrt{x} - 4\log_b \frac{1}{x^7}\right]$$