Questions from Homework

degree is 2 $\to x^2$

⑧ Find a quadratic function f such that

$f(3) = 33 \quad \to \quad \boxed{f(x) = 4x^2 - 2x + 3}$

$f'(3) = 22 \quad \to \quad f'(x) = 8x - 2$

$f''(3) = 8 \qquad f''(x) = 8$

④ $f(x) = \sqrt{1+x^3}$, find $f''(2)$

$$f(x) = (1+x^3)^{1/2}$$

$$f'(x) = \frac{1}{2}(1+x^3)^{-1/2}(3x^2)$$

$$f'(x) = \frac{3x^2}{2(1+x^3)^{1/2}}$$

$$f''(x) = \frac{6x(2)(1+x^3)^{1/2} - 3x^2(1+x^3)^{-1/2}(3x^2)}{\left[2\sqrt{1+x^3}\right]^2}$$

$$f''(x) = \frac{12x(1+x^3)^{1/2} - 9x^4(1+x^3)^{-1/2}}{4(1+x^3)}$$

$$f''(x) = \frac{3x(1+x^3)^{-1/2}\left[\overset{4+4x^3-3x^3}{4(1+x^3) - 3x^3}\right]}{4(1+x^3)}$$

$$f''(x) = \frac{3x(x^3+4)}{4(1+x^3)^{3/2}}$$

$$f''(2) = \frac{3(2)\left[(2)^3+4\right]}{4(1+(2)^3)^{3/2}} = \frac{6(12)}{4(27)} = \frac{72}{108}$$

$$= \frac{2}{3}$$

Implicit Differentiation

So far we have described functions by expressing one variable *explicitly* in terms of another variable: for example,

$$y = x^2$$

$y' = 2x$ or $\frac{dy}{dx} = 2x$

- Sometimes an equation only <u>implicitly</u> defines y as a function (or functions) of x .
- Examples
  - $x^2 + y^2 = 25$ (circle)
  - $x^3 + y^3 = 6xy$

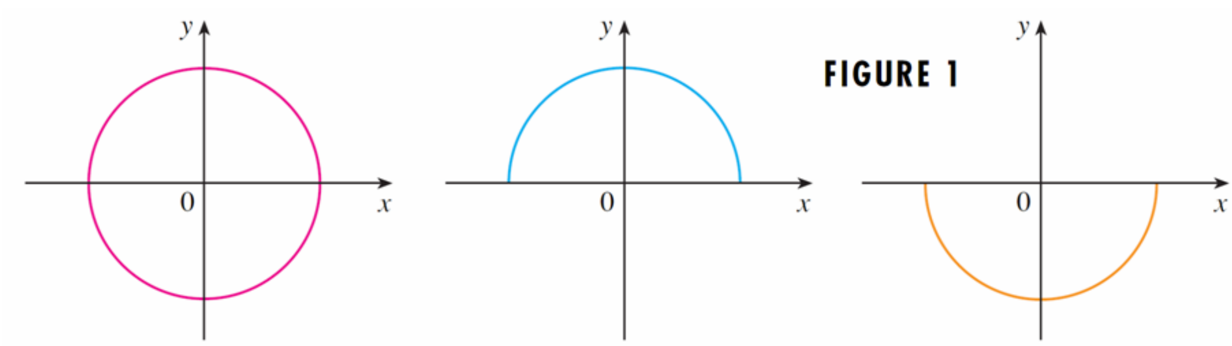
$x^2 + y^2 = 25$

$y^2 = 25 - x^2$

$y = \pm\sqrt{25 - x^2}$

- The first equation could easily be rearranged for $y = \ldots$

$$y = \pm\sqrt{25 - x^2}$$ Actually gives two functions



(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

FIGURE 1

Implicit Differentiation

- There is a way called *implicit differentiation* to find dy/dx without solving for y :
  - First differentiate both sides of the equation with respect to x ;
  - Then solve the resulting equation for y' .
- We will always assume that the given equation does indeed define y as a differentiable function of x .

Example

- For the circle $x^2 + y^2 = 25$, find
  a) dy/dx
  b) an equation of the tangent at the point $(3, 4)$.

Solution:

$x_1 = 3$
$y_1 = 4$

Start by differentiating both sides of the equation:

$$x^2 + y^2 = 25$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

b) (i) $\frac{dy}{dx} = \frac{-x}{y}$

(ii) Find m

$m = \frac{dy}{dx} = -\frac{(3)}{(4)}$

$m = \frac{-3}{4}$

(iii) $y - y_1 = m(x - x_1)$

$y - 4 = \frac{-3}{4}(x - 3)$

$y - 4 = \frac{-3x}{4} + \frac{9}{4}$

$y = \frac{-3}{4}x + \frac{9}{4} + \frac{16}{4}$

$\boxed{y = -\frac{3}{4}x + \frac{25}{4}}$

or $4y = -3x + 25$

$\boxed{3x + 4y - 25 = 0}$

Therefore at the point (3,4) the equation of the tangent would be...

$$y - 4 = -\tfrac{3}{4}(x - 3) \qquad \text{or} \qquad 3x + 4y = 25$$

Same Example Revisited

- Since it is easy to solve this equation for y, we
  - do so, and then
  - find the equation of the tangent line by earlier methods, and then
  - compare the result with our preceding answer:

Solution

- Solving the equation gives $y = \pm\sqrt{25 - x^2}$ as before.
- The point (3, 4) lies on the upper semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$

Differentiate f: (i) $f(x) = (25 - x^2)^{1/2}$

$f'(x) = \frac{1}{2}(25 - x^2)^{-1/2}(-2x)$

$f'(x) = \frac{-x}{(25-x^2)^{1/2}} = \frac{-x}{\sqrt{25-x^2}}$

(ii) Find m:

$f'(3) = \frac{-3}{\sqrt{25-9}} = \frac{-3}{4}$

(iii) $y - y_1 = m(x - x_1)$

Same as before

Solution (cont'd)

- So $f'(3) = -\dfrac{3}{\sqrt{25-3^2}} = -\dfrac{3}{4}$,

  leading to the same equation

  $$3x + 4y = 25$$

  for the tangent that we obtained earlier.

- Note that although this problem could be done both ways, implicit differentiation was easier!

Sometimes Implicit Differentiation is not only the easiest way, it's the *only* way

Example:

$f'g + fg'$

Given $x^3 + y^3 = (6x)(y)$

Find $\frac{dy}{dx}$

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{\cancel{3}(2y - x^2)}{\cancel{3}(y^2 - 2x)}$$

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

$$2x^5 + x^4 y + y^5 = 36$$

Find $\dfrac{dy}{dx}$

Homework

Exercise 2.7 Page 107
Do # 1-3, 5, 7