Questions from Homework

degree is 2 → x^2

⑧ Find a quadratic function f such that

$f(3) = 33 \quad \rightarrow \quad \boxed{f(x) = 4x^2 - 2x + 3}$

$f'(3) = 22 \quad \rightarrow \quad f'(x) = 8x - 2$

$f''(3) = 8 \quad\quad f''(x) = 8$

④ $f(x) = \sqrt{1+x^3}$, find $f''(2)$

$f(x) = (1+x^3)^{1/2}$

$f'(x) = \frac{1}{2}(1+x^3)^{-1/2}(3x^2)$

$f'(x) = \dfrac{3x^2}{2(1+x^3)^{1/2}}$

$f''(x) = \dfrac{6x(2)(1+x^3)^{1/2} - 3x^2(1+x^3)^{-1/2}(3x^2)}{\left[2\sqrt{1+x^3}\right]^2}$

$f''(x) = \dfrac{12x(1+x^3)^{1/2} - 9x^4(1+x^3)^{-1/2}}{4(1+x^3)}$

$4+4x^3-3x^3$

$f''(x) = \dfrac{3x(1+x^3)^{-1/2}\left[4(1+x^3) - 3x^3\right]}{4(1+x^3)}$

$f''(x) = \dfrac{3x(x^3+4)}{4(1+x^3)^{3/2}}$

$f''(2) = \dfrac{3(2)\left[(2)^3+4\right]}{4(1+(2)^3)^{3/2}} = \dfrac{6(12)}{4(27)} = \dfrac{72}{108}$

$= \dfrac{2}{3}$

Implicit Differentiation

So far we have described functions by expressing one variable *explicitly* in terms of another variable: for example,

$$y = x^2$$

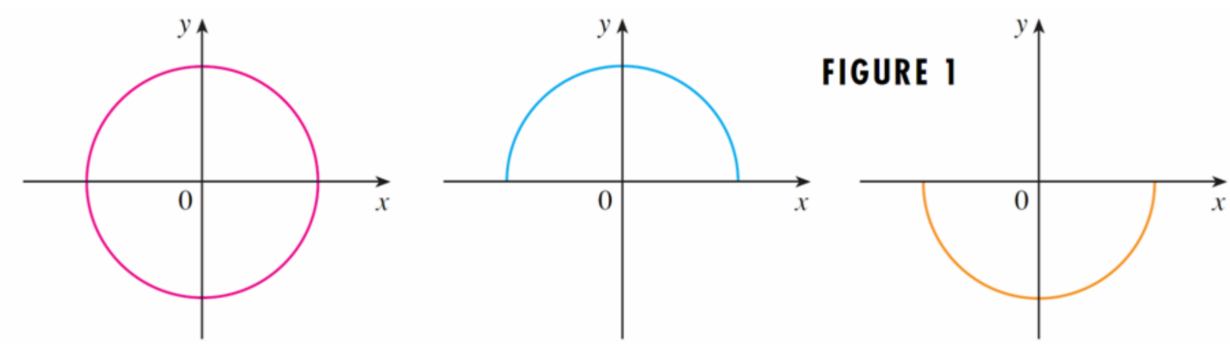
$y' = 2x$ or $\frac{dy}{dx} = 2x$

- Sometimes an equation only <u>implicitly</u> defines y as a function (or functions) of x .
- Examples
  - $x^2 + y^2 = 25$ (circle)
  - $x^3 + y^3 = 6xy$

$$x^2 + y^2 = 25$$
$$y^2 = 25 - x^2$$
$$y = \pm\sqrt{25 - x^2}$$

- The first equation could easily be rearranged for y =...

$$y = \pm\sqrt{25 - x^2}$$ Actually gives two functions



(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

Implicit Differentiation

- There is a way called *implicit differentiation* to find dy/dx without solving for y :
  - First differentiate both sides of the equation with respect to x ;
  - Then solve the resulting equation for y' .
- We will always assume that the given equation does indeed define y as a differentiable function of x .

Example

- For the circle $x^2 + y^2 = 25$, find
  a) dy/dx
  b) an equation of the tangent at the point (3, 4).

Solution:

$x_1 = 3$
$y_1 = 4$

Start by differentiating both sides of the equation:

$$x^2 + y^2 = 25$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

b) (i) $\frac{dy}{dx} = \frac{-x}{y}$

(ii) Find m

$$m = \frac{dy}{dx} = -\frac{(3)}{(4)}$$

$$m = \frac{-3}{4}$$

(iii) $y - y_1 = m(x - x_1)$

$$y - 4 = \frac{-3}{4}(x - 3)$$

$$y - 4 = \frac{-3x}{4} + \frac{9}{4}$$

$$y = \frac{-3}{4}x + \frac{9}{4} + \frac{16}{4}$$

$$\boxed{y = -\frac{3}{4}x + \frac{25}{4}}$$

or $4y = -3x + 25$

$$\boxed{3x + 4y - 25 = 0}$$

Therefore at the point (3,4) the equation of the tangent would be...

$$y - 4 = -\tfrac{3}{4}(x - 3) \qquad \text{or} \qquad 3x + 4y = 25$$

Same Example Revisited

- Since it is easy to solve this equation for y, we
  - do so, and then
  - find the equation of the tangent line by earlier methods, and then
  - compare the result with our preceding answer:

Solution

- Solving the equation gives $y = \pm\sqrt{25 - x^2}$ as before.
- The point (3, 4) lies on the upper semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$

Differentiate f: (i) $f(x) = (25 - x^2)^{1/2}$

$$f'(x) = \frac{1}{2}(25 - x^2)^{-1/2}(-2x)$$

$$f'(x) = \frac{-x}{(25 - x^2)^{1/2}} = \frac{-x}{\sqrt{25 - x^2}}$$

(ii) Find m:

$$f'(3) = \frac{-3}{\sqrt{25-9}} = \frac{-3}{4}$$

(iii) $y - y_1 = m(x - x_1)$

Same as before

Solution (cont'd)

- So $f'(3) = -\dfrac{3}{\sqrt{25-3^2}} = -\dfrac{3}{4}$,

  leading to the same equation

  $$3x + 4y = 25$$

  for the tangent that we obtained earlier.
- Note that although this problem could be done both ways, implicit differentiation was easier!

Sometimes Implicit Differentiation is not only the easiest way, it's the *only* way

Example:

$f'g+fg'$

Given $x^3 + y^3 = 6xy$

Find $\frac{dy}{dx}$

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x} = \frac{\cancel{3}(2y-x^2)}{\cancel{3}(y^2-2x)}$$

$$\frac{dy}{dx} = \frac{2y-x^2}{y^2-2x}$$

with implicit differentiation:

$x \rightarrow 1$

$x^2 \rightarrow 2x$

$y \rightarrow \frac{dy}{dx}$

$y^2 \rightarrow 2y\frac{dy}{dx}$

Find $\frac{dy}{dx}$

$$f'g+fg'$$

$$2(x)^5+(x^4)(y)+(y)^5=36$$

$$10x^4+4x^3y+x^4\frac{dy}{dx}+5y^4\frac{dy}{dx}=0$$

$$x^4\frac{dy}{dx}+5y^4\frac{dy}{dx}=-10x^4-4x^3y$$

$$\frac{dy}{dx}(x^4+5y^4)=-10x^4-4x^3y$$

$$\frac{dy}{dx}=\frac{-10x^4-4x^3y}{x^4+5y^4}$$

$$\boxed{\frac{dy}{dx}=-\frac{10x^4+4x^3y}{x^4+5y^4}}$$

Homework

Exercise 2.7 Page 107
Do # 1-3, 5, 7

Homework

① h) $\dfrac{2x}{x+y} = y$

$$\frac{2(x+y) - 2x\left(1+\frac{dy}{dx}\right)}{(x+y)^2} = \frac{dy}{dx}$$

$$\frac{\cancel{2x} + 2y - \cancel{2x} - 2x\frac{dy}{dx}}{(x+y)^2} = \frac{dy}{dx}$$

$$\frac{2y - 2x\frac{dy}{dx}}{(x+y)^2} = \frac{dy}{dx}$$

$$2y - 2x\frac{dy}{dx} = \frac{dy}{dx}(x+y)^2$$

$$2y = \frac{dy}{dx}(x+y)^2 + 2x\frac{dy}{dx}$$

$$2y = \frac{dy}{dx}\left[(x+y)^2 + 2x\right]$$

$$\boxed{\frac{2y}{(x+y)^2 + 2x} = \frac{dy}{dx}}$$

if $x = 2$ $y = 1$

$$\frac{dy}{dx} = \frac{2}{13}$$

① $\ln^{(x+y)} \frac{2x}{x+y} = y\,(x+y)$???

$$2x = xy + y^2$$

$$2 = 1y + x\frac{dy}{dx} + 2y\frac{dy}{dx}$$

$$2 - y = x\frac{dy}{dx} + 2y\frac{dy}{dx}$$

$$2 - y = \frac{dy}{dx}(x + 2y)$$

$$\boxed{\frac{2-y}{x+2y} = \frac{dy}{dx}}$$

if $x = 2$ $y = 1$

$$\frac{dy}{dx} = \frac{1}{4}$$