Differential and Integral Calculus 120

$\int$ Volumes of Revolution $\int$

Disc Method

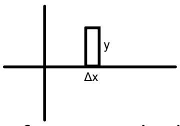
Volumes of Revolution

Another application of a definite integral is its use in finding the volume of a 3-dimensional figure.

Let's suppose we drew a semicircle with its diameter resting on the x-axis, what figure would result if the semicircle was rotated about the x-axis??? sphere

Well, rotating a plane region about any line produces a solid figure.

What solid figure results if we were to rotate a rectangle about the x-axis???



cylinder

the rotation of the rectangle forms a cylindrical disc with height Δx and radius y.

$$\boxed{\begin{array}{l} V = \pi r^2 h \\ V = \pi y^2 \Delta x \end{array}}$$

Similarly (like when we first determined the area under a curve by summing n rectangles and evaluating the sum as a limit as n approaches infinity) , we can determine the volume of a solid figure by summing the volumes of n cylindrical discs as n approaches infinity.

- Finding the volume using this approach is called:

We arrive at the formula for calculating the volume much in the same way we did with calculating the area under a curve using the limit of a Riemann sum.

Again, if we let $y = f(x_i)$ where $x_i = a + i\Delta x$

in this case the radius of i^{th} disc.

and let the height Δx of each cylindrical disc be: $\Delta x = \frac{b-a}{n}$

The volume of the i^{th} cylindrical disc is:

$$V = \pi y^2 \Delta x = \pi[f(x_i)]^2 \Delta x$$

The sum of the volumes of n cylindrical discs becomes:

$$\sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x$$

and the volume of the region from a to b can be written as:

$$V = \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x} = \boxed{\int_a^b \pi[f(x)]^2 dx}$$

Volume of Revolution:

$$\boxed{V = \int_a^b \pi[f(x)]^2 dx}$$

Example:

Find the volume of the solid that is generated when the region under y = 2x from 0 to 2 is rotated about the x-axis.
(the text does solution for region under y = x)

(i) $\Delta x = \frac{b-a}{n}$

$\Delta x = \frac{2-0}{n}$

$\Delta x = \frac{2}{n}$

(ii) $x_i^* = a + i\Delta x$

$x_i^* = 0 + \frac{2i}{n}$

$x_i^* = \frac{2i}{n}$

(iii) $f(x) = 2x$

$f\left(\frac{2i}{n}\right) = 2\left(\frac{2i}{n}\right) = \frac{4i}{n}$

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left[f(x_i^*)\right]^2 \Delta x$$

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left[\frac{4i}{n}\right]^2 \frac{2}{n}$$

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left(\frac{16i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{32\pi i^2}{n^3}$$

Quadratic

$$V = \lim_{n\to\infty} \frac{32\pi}{n^3} \boxed{\sum_{i=1}^{n} i^2}$$

$$V = \lim_{n\to\infty} \frac{\overset{16}{\cancel{32}}\pi}{\cancel{n^3}\,n^2}\left[\frac{\cancel{n}\overbrace{(n+1)(2n+1)}^{2n^2+3n+1}}{\underset{3}{\cancel{6}}}\right]$$

$$V = \lim_{n\to\infty} \frac{32\pi n^2 + 48\pi n + 16\pi}{3n^2}$$

$$\boxed{V = \frac{32\pi}{3}}$$

Example:

Find the volume of the solid that is generated when the region under y = 2x from 0 to 2 is rotated about the x-axis.

(the text does solution for region under y = x)

$$V = \int_a^b \pi [f(x)]^2 dx$$

$$V = \int_0^2 \pi [2x]^2 dx$$

$$V = \pi \int_0^2 4x^2 dx$$

$$V = \pi \left[\frac{4x^3}{3} \right]_0^2$$

$$V = \pi \left[\frac{4(2)^3}{3} - \frac{4(0)^3}{3} \right]$$

$$V = \pi \left[\frac{32}{3} - 0 \right]$$

$$\boxed{V = \frac{32\pi}{3}}$$

Example:

Find the volume of the solid that is generated when the region under y = x^3 from 1 to 2 is rotated about the x-axis. (sketch the region to be rotated)

$$V = \int_a^b \pi [f(x)]^2 dx$$

$$V = \int_1^2 \pi [x^3]^2 dx$$

$$V = \pi \int_1^2 x^6 dx$$

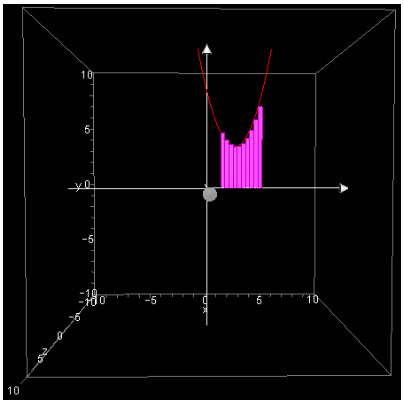
$$V = \pi \left(\frac{x^7}{7}\right)\Big|_1^2$$

$$V = \pi \left[\frac{(2)^7}{7} - \frac{(1)^7}{7}\right]$$

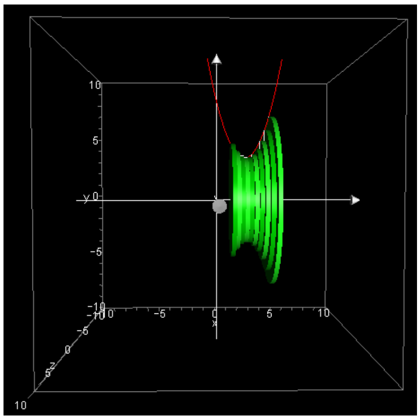
$$V = \pi \left[\frac{128}{7} - \frac{1}{7}\right]$$

$$\boxed{V = \frac{127\pi}{7}}$$

Find the volume of the solid generated by rotating the curve $y = x^2 - 4x + 7$ about the x-axis from $x = 1$ to $x = 4$.



$A = 12$



$$V = \pi \int_1^4 \left(x^2 - 4x + 7\right)^2 dx$$

$$V = \frac{258\pi}{5}$$

$$V = \pi \int_1^4 \left(x^4 - 8x^3 + 30x^2 - 56x + 49\right) dx$$

Homework

Finish review sheet

Solutions to Review

Review

① a) (i) $\Delta x = \frac{2-0}{n} = \frac{2}{n}$ (ii) $x_i^* = 0 + \frac{2i}{n}$ (iii) $f\left(\frac{2i}{n}\right) = \left(\frac{2i}{n}\right)^2 + 2$

$x_i^* = \frac{2i}{n}$ $= \frac{4i^2}{n^2} + 2$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i^2}{n^2} + 2\right)\frac{2}{n}$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{8i^2}{n^3} + \frac{4}{n}$$

$$A = \lim_{n\to\infty} \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} \frac{4}{n}$$

$$A = \lim_{n\to\infty} \frac{\cancel{8}^{\,4}}{n^{\cancel{3}\,2}}\left[\frac{\cancel{n}\overbrace{(n+1)(2n+1)}^{2n^2+3n+1}}{\cancel{6}\,3}\right] + \frac{4}{\cancel{n}}\cdot\cancel{n}$$

$$A = \lim_{n\to\infty} \frac{8n^2+12n+4}{3n^2} + 4$$

$$A = \frac{8}{3} + 4 = \frac{8}{3} + \frac{12}{3} = \boxed{\frac{20}{3}}$$

b) (i) $\Delta x = \frac{3-1}{n} = \frac{2}{n}$ (ii) $x_i^* = 1 + \frac{2i}{n}$ (iii) $f\left(1+\frac{2i}{n}\right) = 3\left(1+\frac{2i}{n}\right)^2 + 6\left(1+\frac{2i}{n}\right) - 2$

$= 3\left[1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right] + 6 + \frac{12i}{n} - 2$

$= 3 + \frac{12i}{n} + \frac{12i^2}{n^2} + 6 + \frac{12i}{n} - 2$

$= \frac{12i^2}{n^2} + \frac{24i}{n} + 7$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{12i^2}{n^2} + \frac{24i}{n} + 7\right)\frac{2}{n}$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{24i^2}{n^3} + \frac{48i}{n^2} + \frac{14}{n}\right)$

$A = \lim_{n\to\infty} \frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{48}{n^2}\sum_{i=1}^{n} i + \sum_{i=1}^{n} \frac{14}{n}$

$A = \lim_{n\to\infty} \frac{\overset{4}{\cancel{24}}}{\cancel{n^3}}\left[\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}}\right] + \frac{48}{\cancel{n^2}}\left[\frac{\cancel{n}(n+1)}{2}\right] + \frac{14}{\cancel{n}} \cdot \cancel{n}$

$A = \lim_{n\to\infty} \frac{4(2n^2+3n+1)}{n^2} + \frac{48(n+1)}{2n} + 14$

$A = \lim_{n\to\infty} \frac{8n^2+12n+4}{n^2} + \frac{48n+48}{2n} + 14$

$A = 8 + 24 + 14 = \boxed{46}$

② a) $A = \int_0^2 (x^2+2)\,dx$

$A = \frac{x^3}{3} + 2x \Big|_0^2$

$A = \frac{(2)^3}{3} + 2(2) - \left[\frac{(0)^3}{3} + 2(0)\right]$

$A = \frac{8}{3} + 4 - 0$

$A = \frac{8}{3} + \frac{12}{3} = \boxed{\frac{20}{3}}$

b) $A = \int_1^3 (3x^2+6x-2)\,dx$

$A = x^3 + 3x^2 - 2x \Big|_1^3$

$A = (3)^3 + 3(3)^2 - 2(3) - [(1)^3 + 3(1)^2 - 2(1$

$A = 27 + 27 - 6 - [1 + 3 - 2]$

$A = 48 - 2$

$\boxed{A = 46}$

③ a) $f(x) = 6x^2 + \csc^2 x$

$F(x) = \frac{6x^3}{3} - \cot x + C$

$\boxed{F(x) = 2x^3 - \cot x + C}$

b) $f(x) = 10x^4 - 2\sec^2 x$

$F(x) = \frac{10x^5}{5} - 2\tan x + C$

$\boxed{F(x) = 2x^5 - 2\tan x + C}$

c) $\int \left(12x^2 - \sqrt{x^3} + \sin(5x) + \frac{2}{x}\right) dx$

$= \int \left(12x^2 - x^{3/2} + \sin(5x) + 2 \cdot \frac{1}{x}\right) dx$

$= \frac{12x^3}{3} - \frac{x^{5/2}}{5/2} - \frac{1}{5}\cos(5x) + 2\ln|x| + C$

$\boxed{= 4x^3 - \frac{2}{5}x^{5/2} - \frac{1}{5}\cos(5x) + 2\ln|x| + C}$

d) $\int \frac{x}{\sqrt{4+x^2}}\, dx = \int u^{-1/2} \cdot \frac{1}{2} du$

$u = 4 + x^2$

$du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$= \frac{1}{2}\int u^{-1/2} du$

$= \frac{1}{2}\left(2u^{1/2}\right) + C$

$= u^{1/2} + C$

$= (4+x^2)^{1/2} + C$

$\boxed{= \sqrt{4+x^2} + C}$

e) $\int \sqrt{x}\, \underbrace{\ln x}_{u}\, dx = \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{3/2} \cdot \frac{1}{x}\, dx$

$u = \ln x \qquad dv = x^{1/2} dx$

$du = \frac{1}{x} dx \qquad v = \frac{2}{3}x^{3/2}$

$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2} dx$

$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\left(\frac{2}{3}x^{3/2}\right) + C$

$\boxed{= \frac{2}{3}x^{3/2}\ln^{x} - \frac{4}{9}x^{3/2} + C}$

Review

4 a) $y = \frac{1}{x}$ from 1 to 4

$$V = \int_1^4 \pi (f(x))^2\, dx$$

$$V = \int_1^4 \pi \left(\frac{1}{x}\right)^2 dx$$

$$V = \pi \int_1^4 x^{-2}\, dx$$

$$V = \pi \Big[-x^{-1}\Big]_1^4$$

$$V = \pi \left[-\frac{1}{x}\right]_1^4$$

$$V = \pi \left[-\frac{1}{4} + \frac{1}{1}\right]$$

$$V = \pi \left[-\frac{1}{4} + \frac{4}{4}\right] = \boxed{\frac{3\pi}{4}}$$

b) $f(x) = x^2 + 1$ from 1 to 3

$$V = \int_1^3 \pi [f(x)^2]\, dx$$

$$V = \int_1^3 \pi [x^2 + 1]^2\, dx$$

$$V = \pi \int_1^3 x^4 + 2x^2 + 1\, dx$$

$$V = \pi \left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_1^3$$

$$V = \pi \left[\frac{(3)^5}{5} + \frac{2(3)^3}{3} + 3 - \left(\frac{(1)^5}{5} + \frac{2(1)^3}{3} + 1\right)\right]$$

$$V = \pi \left[\frac{243}{5} + \frac{54}{3} + 3 - \frac{1}{5} - \frac{2}{3} - 1\right]$$

$$V = \pi \left[\frac{242}{5} + \frac{52}{3} + 2\right]$$

$$V = \pi \left[\frac{726}{15} + \frac{260}{15} + \frac{30}{15}\right] = \boxed{\frac{1016\pi}{15}}$$

L 15

5a) $\int_0^{\pi/2} x\sin x\,dx = -x\cos x\Big|_0^{\pi/2} - \int_0^{\pi/2} -\cos x\,dx$

$u = x \quad dv = \sin x\,dx$

$du = dx \quad v = -\cos x$

$= -x\cos x\Big|_0^{\pi/2} + \int_0^{\pi/2} \cos x\,dx$

$= -x\cos x\Big|_0^{\pi/2} + \sin x\Big|_0^{\pi/2}$

$= -\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + 0\cos(0) + \sin\left(\frac{\pi}{2}\right) - \sin(0)$

$= \frac{-\pi}{2}(0) + 0(1) + 1 - 0$

$= 0 + 0 + 1 - 0 = \boxed{1}$

5 b) $\int_0^1 \frac{1}{(3x+2)^2}\,dx = \int_2^5 u^{-2}\cdot\frac{1}{3}du = \frac{1}{3}\int_2^5 u^{-2}\,du$

$u=3x+2$

$du=3dx$

$\frac{1}{3}du=dx$

x	u
1	5
0	2

$= \frac{1}{3}\left[-u^{-1}\right]_2^5$

$= \frac{1}{3}\left[-\frac{1}{u}\right]_2^5$

$= -\frac{1}{3u}\Big|_2^5$

$= -\frac{1}{3(5)} + \frac{1}{3(2)}$

$= -\frac{1}{15} + \frac{1}{6}$

$= \frac{-2}{30} + \frac{5}{30} = \frac{3}{30} = \boxed{\frac{1}{10}}$

6) $y=6-x^2$ (top) and $y=3-2x$ (bottom) $\quad A=\int_{-1}^{3}(6-x^2)-(3-2x)\,dx$

top / Bottom

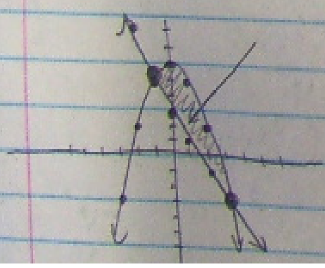
6) $y = 6 - x^2$ and $y = 3 - 2x$

$$A = \int_{-1}^{3} (6 - x^2) - (3 - 2x)\, dx$$

1) Quick Sketch

$$A = \int_{-1}^{3} -x^2 + 2x + 3\, dx$$

$y = 6 - x^2$		$y = 3 - 2x$	
x	y	x	y
-2	2	-2	7
→ -1	5	-1	5
0	6	0	3
1	5	1	1
2	2	2	-1
→ 3	-3	3	-3



$$A = -\frac{x^3}{3} + x^2 + 3x \Big|_{-1}^{3}$$

$$A = -\frac{(3)^3}{3} + (3)^2 + 3(3) - \left[-\frac{(1)^3}{3} + (-1)^2 + 3(-1)\right]$$

$$A = -9 + 9 + 9 - \left[-\frac{1}{3} + 1 - 3\right]$$

$$A = 9 + \frac{1}{3} - 1 + 3$$

$$A = 11 + \frac{1}{3} = \frac{33}{3} + \frac{1}{3} = \boxed{\frac{34}{3}}$$

Example:

Find the volume of the solid that is generated when the region between y = sin x and y = cos x from 0 to π/4 is rotated about the x-axis.

(sketch the region to be rotated)

Is a subtraction of some amount necessary??

Hint: double angle identity may help.

$$V = \frac{\pi}{2}$$