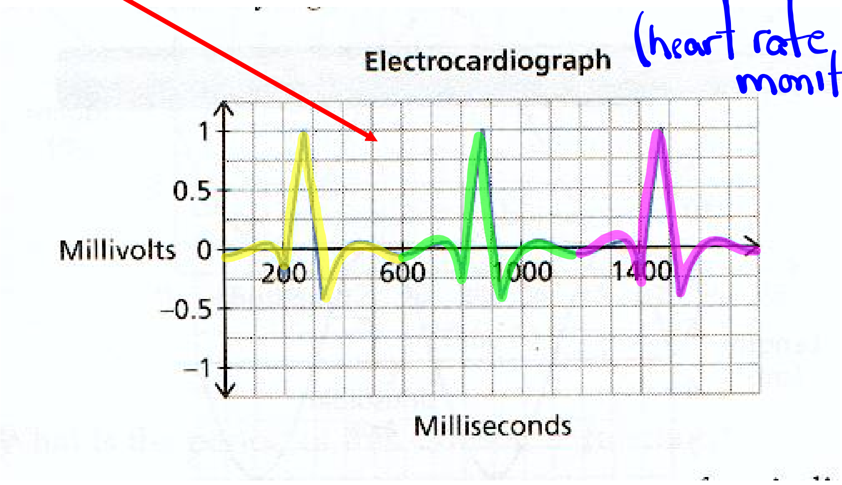
# Sinusoidal Relations (Trig Graphs)

Periodic Function: A function for which the dependent variable takes on the same set of values over and over again as the independent variable changes.

(a function that repeats)

Example of periodic behavior

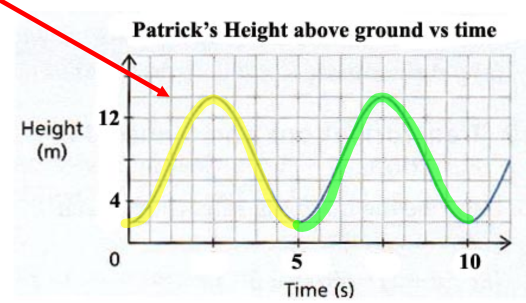
(heart rate monitor)



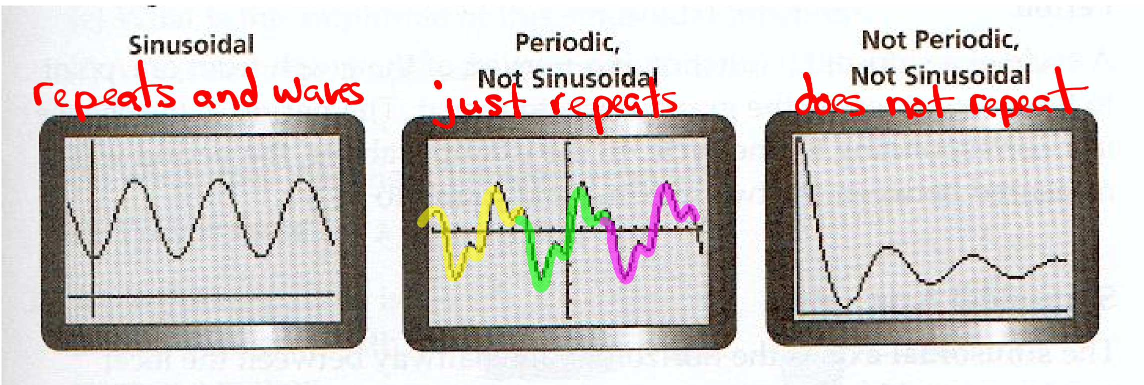

Sinusoidal Function: A periodic function that looks like waves, where any portion of the curve can be translated onto another portion of the curve.

(Repeats and looks like a smooth wave).

Example of sinusoidal behavior




These illustrations should summarize periodic and sinusoidal...

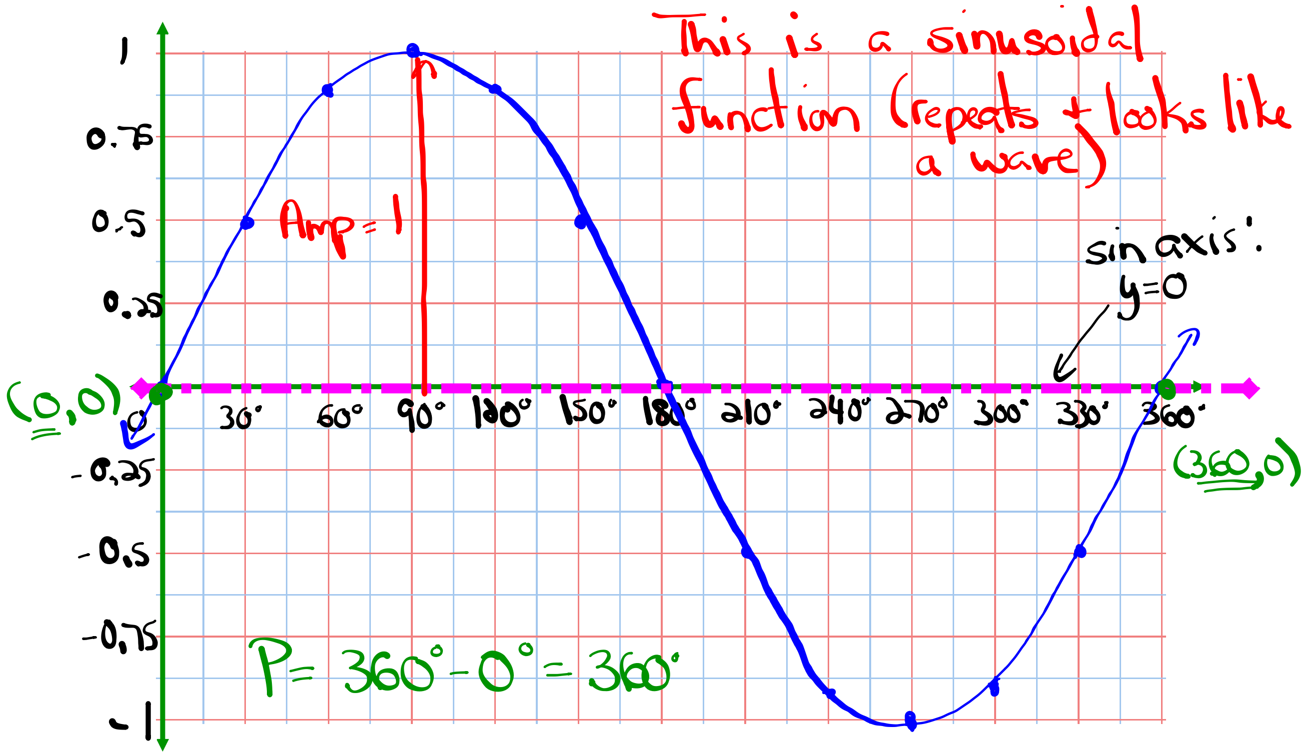


Let's examine the graph of $y = \sin\theta$

$y = \sin x$

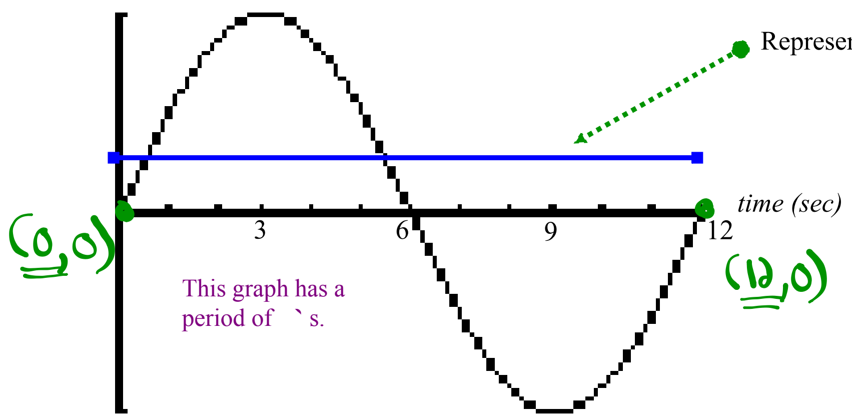
| $\theta$ | 0 | 30 | 60 | 90 | 120 | 150 | 180 | 210 | 240 | 270 | 300 | 330 | 360 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0 | 0.5 | 0.87 | 1 | 0.87 | 0.5 | 0 | -0.5 | -0.87 | -1 | -0.87 | -0.5 | 0 |

Now plot the above points...



# Vocabulary of Sinusoidal Functions

## I. Period:
The change in $x$ corresponding to one cycle. (one repitition)




$P = 12 - 0 = 12$

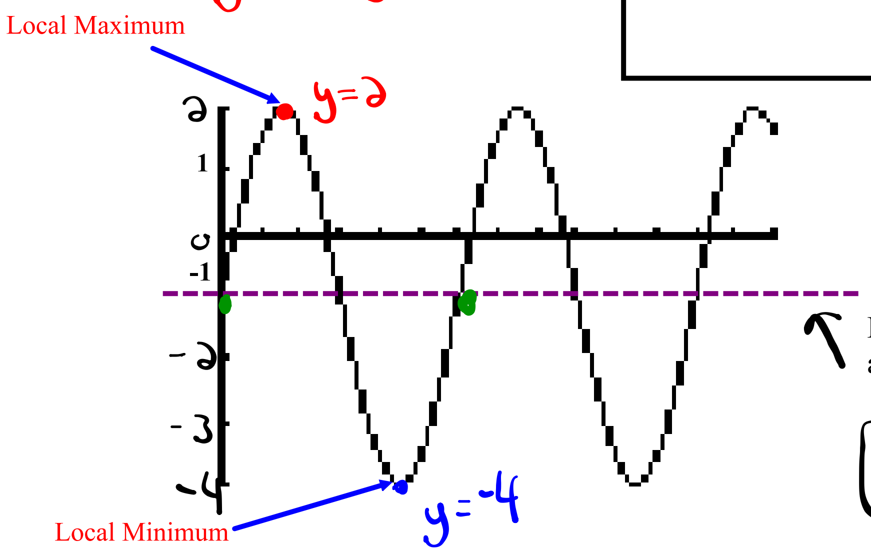
## II. Sinusoidal Axis:
The horizontal line halfway between the local maximum and local minimum.

highest (y) lowest (y)

$$\text{Sinusoidal Axis} = \frac{\text{Max} + \text{Min}}{2}$$

$= \frac{2 + (-4)}{2}$

$= \frac{-2}{2} = -1$




Sinusoidal Axis

Equation of sinusoidal axis is $y = -1$

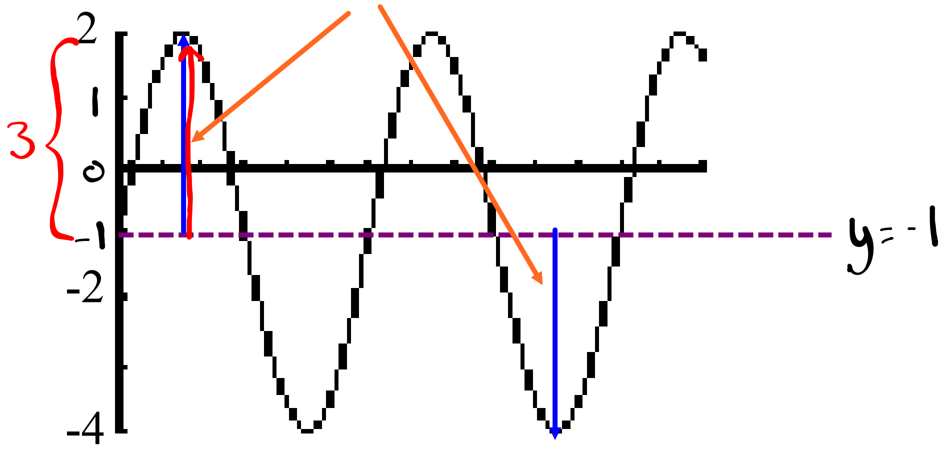
$\boxed{k = -1}$

## III. Amplitude:

The vertical distance from the sinusoidal axis to a local maximum or local minimum. Amplitude = $|a|$

Amp is always positive

Amplitude would equal 3



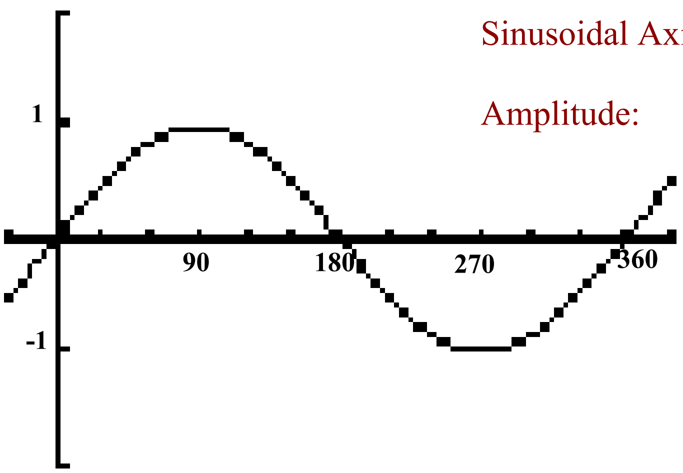

# Summarize...

Here is the graph of $y = \sin\theta$

Period :

Sinusoidal Axis:

Amplitude:

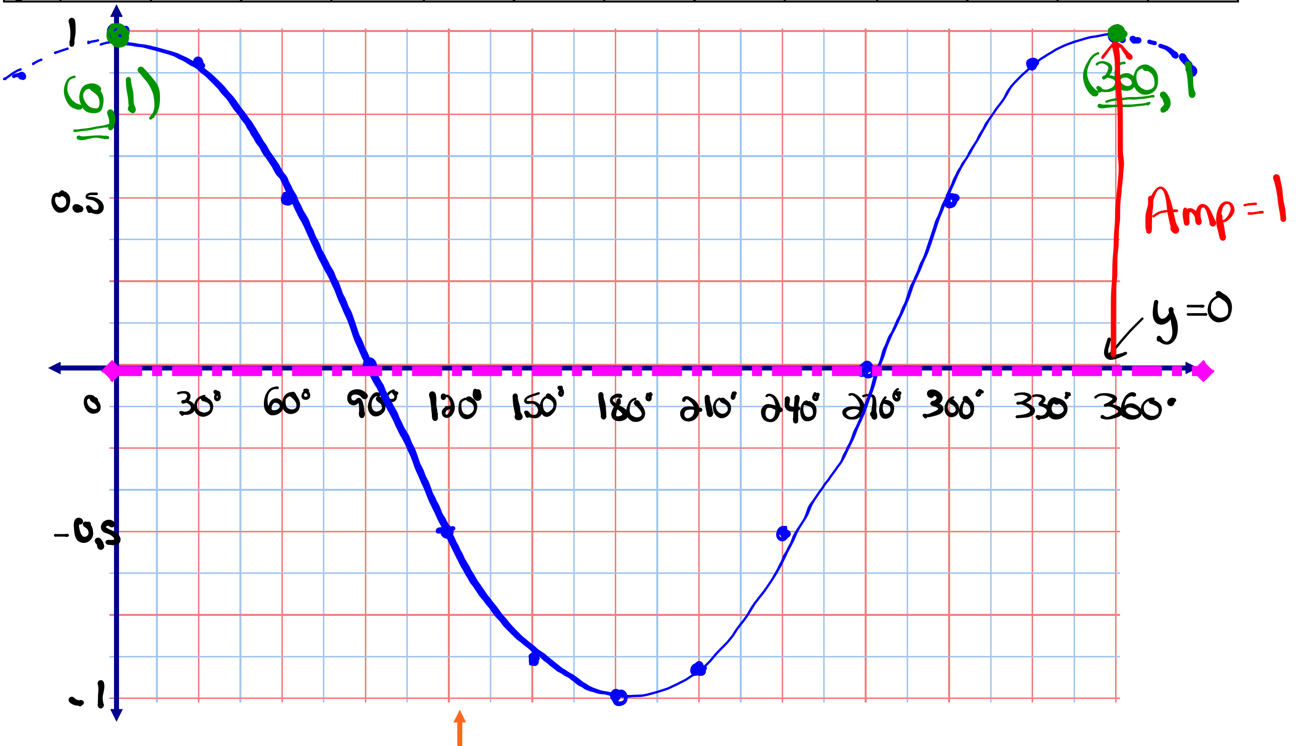


# What about $y = \cos\theta$ ?

$y = \cos x$

Complete the table of values and sketch below

| $\theta$ | 0 | 30 | 60 | 90 | 120 | 150 | 180 | 210 | 240 | 270 | 300 | 330 | 360 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 1 | 0.87 | 0.5 | 0 | -0.5 | -0.87 | -1 | -0.87 | -0.5 | 0 | 0.5 | 0.87 | 1 |




Is this a sinusoidal function? Yes (Repeats + looks like waves)

What about the period, sinusoidal axis, and amplitude?

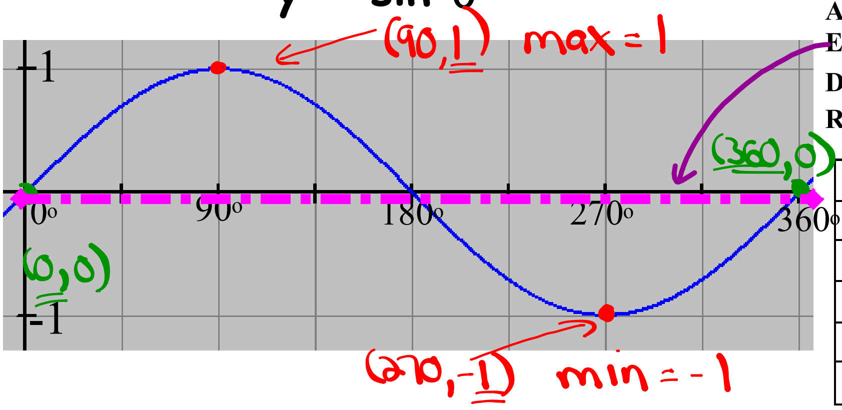
Period = $360° - 0° = 360°$

sinusoidal axis = $\frac{\text{Max} + \text{Min}}{2} = \frac{1 + (-1)}{2} = \frac{0}{2} = 0$ $(y = 0)$

Amplitude = 1

## Basic Trig Graphs (Base Functions)

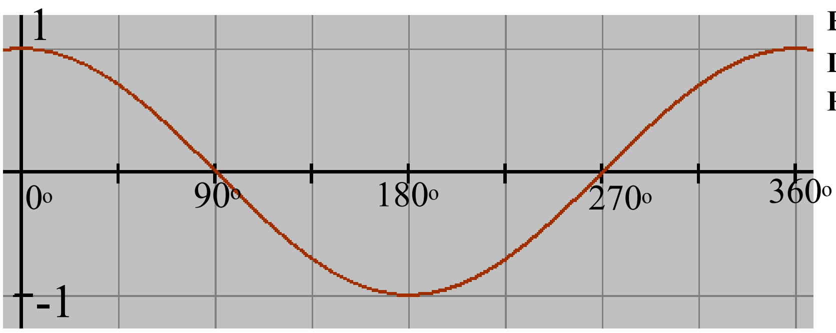
### y = sin θ




**Period = 360°**
**Amplitude = 1**
**Eq'n of Sinusoidal Axis: y = 0**
**Domain: {θ ∈ R }**
**Range: { -1 ≤ y ≤ 1 }**

| θ | y |
|---|---|
| 0° | 0 |
| 90° | 1 |
| 180° | 0 |
| 270° | −1 |
| 360° | 0 |

### y = cos θ




**Period = 360°**
**Amplitude = 1**
**Eq'n of Sinusoidal Axis: y = 0**
**Domain: {θ ∈ R }**
**Range: { -1 ≤ y ≤ 1 }**

| θ | y |
|---|---|
| 0° | 1 |
| 90° | 0 |
| 180° | −1 |
| 270° | 0 |
| 360° | 1 |

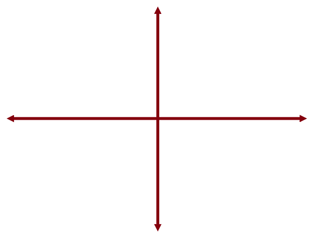
## Transformations of the Sinusoidal Function

Recall...

$$y = -2(x-3)^2 + 4$$

Vertex $\Rightarrow$

Sketch $\Rightarrow$



Now, let's look at a sinusoidal function...

$$y = -2\sin[3(\theta - 60°)] - 1$$

Amp = 2

$a=-2 \rightarrow$ A vertical stretch by a factor of 2 and a reflection in the x-axis

$b=3 \rightarrow$ A horizontal stretch by a factor of $\frac{1}{3}$

$h=60° \rightarrow$ A horizontal translation of 60° right.

$k=-1 \rightarrow$ A vertical translation 1 unit down.

equation of sinusoidal axis: $y=-1$

Mapping Rule: $(x,y) \rightarrow \left[\frac{1}{b}x+h,\ ay+k\right]$

$(x,y) \rightarrow \left[\frac{1}{3}x+60°,\ -2y-1\right]$

(base) $y=\sin x$ $\rightarrow$ $y=-2\sin[3(x-60°)]-1$

| x | y |
|---|---|
| 0 | 0 |
| 90° | 1 |
| 180° | 0 |
| 270° | -1 |
| 360° | 0 |

| x | y |
|---|---|
| 60° | -1 |
| 90° | -3 |
| 120° | -1 |
| 150° | 1 |
| 180° | -1 |

## Equations in Standard Form

$$y = a\sin\left[b(x-c)\right]+d \quad \text{or} \quad y = a\cos[b(x-h)]+k$$

$a$ = ***Amplitude*** → influences how tall the sine curve is. (always positive)

$b = \dfrac{360^\circ}{P}$ → influences how often the pattern repeats. $\left(P = \dfrac{360^\circ}{b}\right)$

$P$ ← Period

$c$ = ***Horizontal Translation*** → Influences how far to the left or the right that the graph will shift. (Phase Shift)

- If C is positive → Shift Left
- If C is negative → Shift Right

} Inside Brackets

$d$ = ***Vertical Translation*** → influences how far up and down the graph will shift.

- If $d$ is positive → Shift Up
- If $d$ is negative → Shift Down
- equal to the sinusoidal axis:
  ↳ equation of sinusoidal axis: $y = d$

Example:

$$2y + 5 = -6\sin\left(\tfrac{1}{3}x - 30^\circ\right) - 3 \quad \text{(Subtract 5 from both sides)}$$

$$\frac{2y}{2} = \frac{-6\sin\left(\tfrac{1}{3}x - 30^\circ\right)}{2} - \frac{8}{2} \quad \text{(Divide by 2)}$$

$$y = -3\sin\left(\tfrac{1}{3}x - 30^\circ\right) - 4 \quad \text{(Factor out a } \tfrac{1}{3}\text{)}$$

$$y = -3\sin\left[\tfrac{1}{3}(x - 90^\circ)\right] - 4$$

$a = -3 \qquad b = \frac{1}{3} \qquad h = 90^\circ \qquad k = -4$

$\text{Amp} = 3 \qquad P = \dfrac{360^\circ}{\frac{1}{3}} = 1080^\circ$

equation of sinusoidal axis: $y = -4$

$$b = \frac{360^\circ}{P} \text{ or } \frac{2\pi}{P}$$

$$P = \frac{360^\circ}{b} \text{ or } \frac{2\pi}{b}$$

h) $2 \cdot \quad \frac{1}{2}(y+2) \overset{2\cdot}{=} 3\cos(x-90^\circ) + 0 \cdot 2$ $(a+k)$

$$y+2 = 6\cos(x-90^\circ)$$

$$y = \underline{\underline{6}}\cos(x-\underline{\underline{90^\circ}})-\underline{\underline{2}}$$

$a = 6$ $h = 90^\circ$ equation of sinusoidal axis: $y=-2$

$b = 1$ $k = -2$ $P = \frac{360^\circ}{b} = \frac{360^\circ}{1} = 360^\circ$

# Homework

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ex: $2y - 5 \overset{+5}{=} -4\cos[3x - 90^\circ] - 7^{+5}$

$$\frac{2y}{2} = \frac{-4}{2}\cos[3x - 90^\circ] \frac{-2}{2}$$

Factor

$$y = -2\cos[3x - 90^\circ] - 1$$

$$y = \underline{\underline{-2}}\cos[\underline{\underline{3}}(x - \underline{\underline{30^\circ}})] \underline{\underline{-1}}$$

$a = -2$ (Amp = 2) vertically stretched by a factor of 2 and reflected in x-axis

$b = 3$ horizontally stretched by a factor of $\frac{1}{3}$

$h = 30^\circ$ translated $30^\circ$ right

$K = -1$ " 1 unit down

$y = a\cos[b(x-h)] + k$

① d) $y - 5 = 6\cos\left[\frac{1}{3}\left(x - \frac{\pi}{2}\right)\right] - 2$

$y = 6\cos\left[\frac{1}{3}\left(x - \frac{\pi}{2}\right)\right] + 3$

$a = 6$ $h = \frac{\pi}{2}$ equation of sin. axis: $y = 3$

$b = \frac{1}{3}$ $k = 3$ $P = \frac{2\pi}{b} = 2\pi \div \frac{1}{3} = 2\pi \cdot \frac{3}{1} = 6\pi$

g) $y + 5 = -2\sin\left(4x + \frac{\pi}{3}\right)$

$y = -2\sin\left(4x + \frac{\pi}{3}\right) - 5$ (Factor out a 4)

$y = -2\sin\left[4\left(x + \frac{\pi}{12}\right)\right] - 5$

$\frac{\pi}{3} \div 4$

$\frac{\pi}{3} \times \frac{1}{4} = \frac{\pi}{12}$

$a = -2$ $h = -\frac{\pi}{12}$ equation of sin. axis: $y = -5$

$b = 4$ $k = -5$ $P = \frac{2\pi}{b} = \frac{2\pi}{4} = \frac{\pi}{2}$

## Sketching Sinusoidal Functions using Transformations

Development of a standard form for sinusoidal functions...

Standard Form ⟶ $y = a\sin[b(x-h)]+k$

1. Reflection: If a < 0 the graph will be reflected in the *x*-axis.

2. Amplitude: The amplitude of the graph will be equal to | a | .

3. Period: The period of the graph will be equal to $\frac{360°}{b}$ or $\frac{2\pi}{b}$

4. Horizontal Phase Shift: The graph will shift "h" units to the right/left. (Translation)

5. Vertical Translation: The graph will shift "k" units up/down.

## Transformations of Sinusoidal Functions

$a=-2$ $b=3$ $P=\frac{360°}{3}=120°$ $h=-30°$ $k=-2$

Example: $f(\theta) = -2\sin 3(\theta + 30°) - 2$

| | |
|---|---|
| **Domain** | $\{\theta \mid \theta \in \mathbb{R}\}$ or $(-\infty,\infty)$ |
| **Range** | $\{y \mid -4 \le y \le 0, y \in \mathbb{R}\}$ or $[-4,0]$ |
| **Reflection** | in the x-axis |
| **Amplitude** | 2 |
| **Horizontal Phase Shift** | 30° left |
| **Vertical Translation** | 2 down |
| **Period** | 120° |

max = k + Amp. = −2 + 2 = 0

min = k − Amp = −2 − 2 = −4

State ***a, b, h, k, and P*** from the following sinusoidal equations:

$$2y+6=4\sin(4x+\frac{\pi}{2})-2$$

$$\frac{2y}{2}=\frac{4\sin(4x+\frac{\pi}{2})}{2}-\frac{8}{2}$$ (Divide a+k)

$$y=2\sin(4x+\frac{\pi}{2})-4$$ (Factor out a 4)

$$y=2\sin[4(x+\frac{\pi}{8})]-4$$

$\frac{\pi}{2}\div 4$

$\frac{\pi}{2}\times\frac{1}{4}=\frac{\pi}{8}$

$$y=2\sin[4(x+\frac{\pi}{8})]-4$$

$a=2$ $b=4$ $h=-\frac{\pi}{8}$ $k=-4$ $P=\frac{2\pi}{b}=\frac{2\pi}{4}=\frac{\pi}{2}$

equation of sin axis: $y=-4$

This time we will graph the same function using a mapping:

$$y = \underline{3}\cos[\underline{2}(\theta - 135°)] + 2$$

$a = 3$ $b = 2$ $h = 135°$ $k = 2$

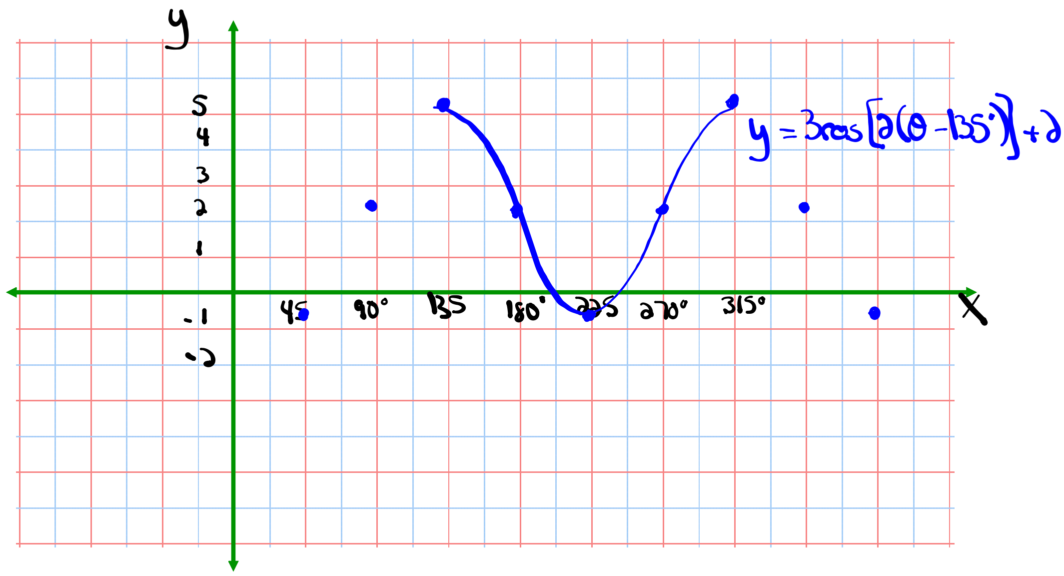
$$(\theta, y) \rightarrow \left[\frac{1}{2}\theta + 135°,\ 3y + 2\right]$$

$y = \cos\theta$

| $\theta$ | $y$ |
|---|---|
| 0 | 1 |
| 90 | 0 |
| 180 | -1 |
| 270 | 0 |
| 360 | 1 |

New points after mapping →

| $\theta$ | $y$ |
|---|---|
| 135° | 5 |
| 180° | 2 |
| 225° | -1 |
| 270° | 2 |
| 315° | 5 |




| | |
|---|---|
| DOMAIN | $\{\theta \mid \theta \in R\}$ or $(-\infty, \infty)$ |
| RANGE | $\{y \mid -1 \le y \le 5, y \in R\}$ or $[-1, 5]$ |
| AMPLITUDE | 3 |
| PERIOD | 180° |
| PHASE SHIFT | 135° right |
| VERTICAL TRANSLATION | 2 up |
| EQUATION OF SINUSOIDAL AXIS | $y = 2$ |

$P = \frac{360°}{2} = 180°$

# Use Mapping to Graph

$$\frac{3y}{3}=\frac{-6}{3}\cos(3x-\pi)-\frac{9}{3}$$

$y=-2\cos(3x-\pi)-3$ (Factor out a 3)

$y=-2\cos\left[3\left(x-\frac{\pi}{3}\right)\right]-3$

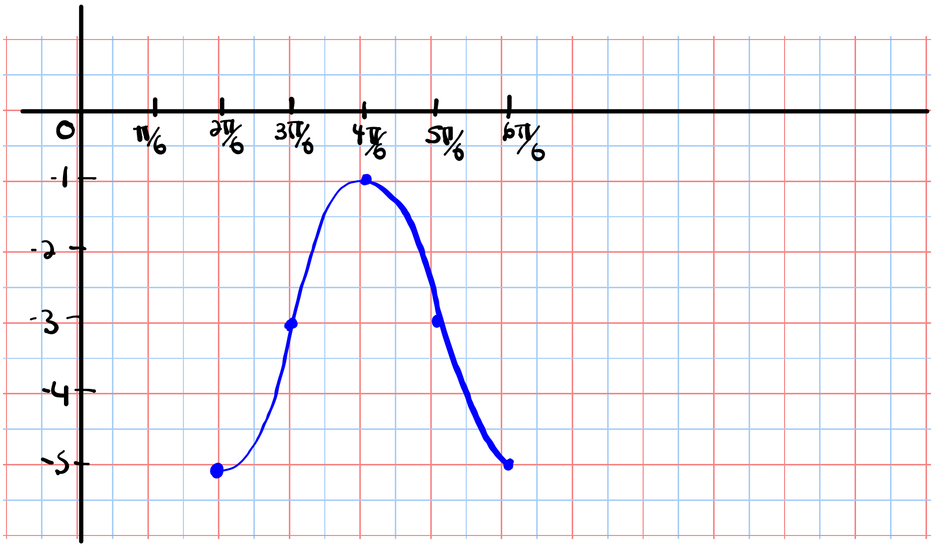
$a=-2$ $b=3$ $h=\frac{\pi}{3}$ $k=-3$ $P=\frac{2\pi}{b}=\frac{2\pi}{3}$

$y=\cos x$ $(x,y)\rightarrow\left[\frac{1}{3}x+\frac{\pi}{3},\ -2y-3\right]$

| $\theta$ | $y$ |
|---|---|
| 0 | 1 |
| $\pi/2$ | 0 |
| $\pi$ | -1 |
| $3\pi/2$ | 0 |
| $2\pi$ | 1 |

New points after mapping

| | $\theta$ | $y$ |
|---|---|---|
| $2\pi/6$ | $\pi/3$ | -5 |
| $3\pi/6$ | $\pi/2$ | -3 |
| $4\pi/6$ | $2\pi/3$ | -1 |
| $5\pi/6$ | $5\pi/6$ | -3 |
| $6\pi/6$ | $\pi$ | -5 |




| | |
|---|---|
| Domain | $\{x \mid x\in R\}$ or $(-\infty,\infty)$ |
| Range | $\{y \mid -5\le y\le -1,\ y\in R\}$ or $[-5,-1]$ |
| Amplitude | 2 |
| Period | $2\pi/3$ |
| Phase Shift | $\pi/3$ right |
| Vertical Translation | 3 down |
| Equation of Sinusoidal Axis | $y=-3$ |

$$\frac{1}{3}x + \frac{\pi}{3}$$

if $x=0 \rightarrow \frac{1}{3}(0)+\frac{\pi}{3} = \frac{\pi}{3} = \frac{2\pi}{6}$

if $x=\frac{\pi}{2} \rightarrow \frac{1}{3}\left(\frac{\pi}{2}\right)+\frac{\pi}{3} = \frac{\pi}{6}+\frac{2\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2} = \frac{3\pi}{6}$

if $x=\pi \rightarrow \frac{1}{3}(\pi)+\frac{\pi}{3} = \frac{\pi}{3}+\frac{\pi}{3} = \frac{2\pi}{3} = \frac{4\pi}{6}$

if $x=\frac{3\pi}{2} \rightarrow \frac{1}{3}\left(\frac{3\pi}{2}\right)+\frac{\pi}{3} = \frac{3\pi}{6}+\frac{2\pi}{6} = \frac{5\pi}{6} = \frac{5\pi}{6}$

if $x=2\pi \rightarrow \frac{1}{3}(2\pi)+\frac{\pi}{3} = \frac{2\pi}{3}+\frac{\pi}{3} = \frac{3\pi}{3} = \pi = \frac{6\pi}{6}$

# Example...

Graph the equation $y = -3\sin(2\theta + \pi) + 1$ using mapping notation.

$y = -3\sin\left[2\left(\theta + \frac{\pi}{2}\right)\right] + 1$

$a = -3$ $b = 2$ $h = -\frac{\pi}{2}$ $k = 1$

| | |
|---|---|
| AMPLITUDE | 3 |
| PERIOD | $\frac{2\pi}{2} = \pi$ |
| PHASE SHIFT | $\pi/2$ (Left) |
| VERTICAL TRANSLATION | 1 (up) |
| EQUATION OF SINUSOIDAL AXIS | $y = 1$ |

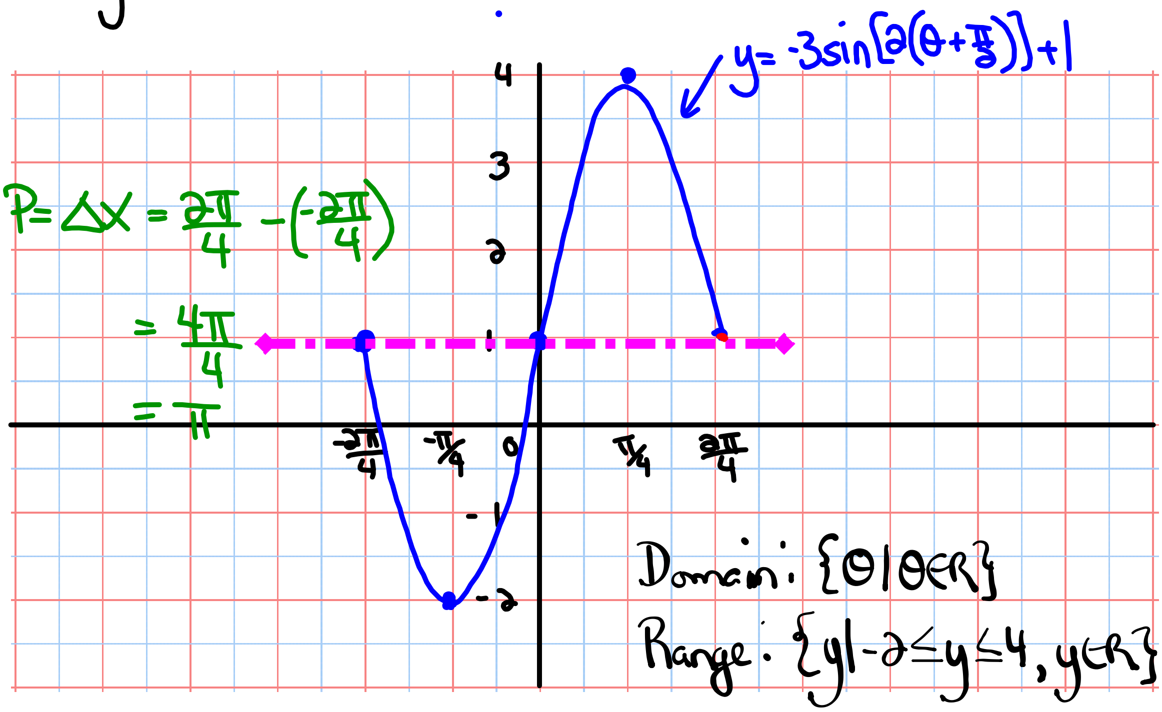
$y = \sin\theta$

| $\theta$ | $y$ |
|---|---|
| 0 | 0 |
| $\pi/2$ | 1 |
| $\pi$ | 0 |
| $3\pi/2$ | -1 |
| $2\pi$ | 0 |

$(x, y) \rightarrow \left[\frac{1}{2}x - \frac{\pi}{2}, -3y + 1\right]$

New points after mapping

| $\theta$ | $y$ |
|---|---|
| $-\pi/2$ | 1 |
| $-\pi/4$ | -2 |
| 0 | 1 |
| $\pi/4$ | 4 |
| $\pi/2$ | 1 |




$P = \Delta x = \frac{2\pi}{4} - \left(-\frac{2\pi}{4}\right)$

$= \frac{4\pi}{4}$

$= \pi$

Domain: $\{\theta \mid \theta \in R\}$

Range: $\{y \mid -2 \leq y \leq 4, y \in R\}$

Hopefully you are not too puzzled for this one...

2\. $\frac{1}{2}(y+1)=3\cos\left(\frac{1}{2}\theta-90°\right)+2$ (a+k)

$y+1=6\cos\left[\frac{1}{2}\theta-90°\right]+4$

$y=6\cos\left[\frac{1}{2}\theta-90°\right]+3$ (factor out ½)

$y=6\cos\left[\frac{1}{2}(\theta-180°)\right]+3$

$90\div\frac{1}{2}$

$90\times2=180°$

$a=6$ $b=\frac{1}{2}$ $h=180°$ $k=3$

Mapping:

$(x,y)\rightarrow[2x+180°, 6y+3]$

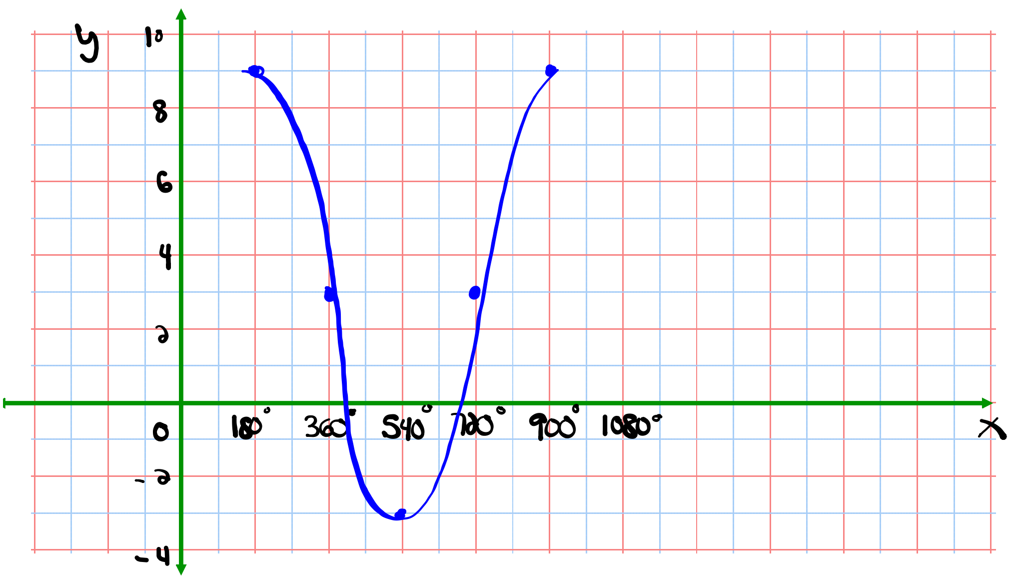
$y=\cos x$

| $\theta$ | $y$ |
|---|---|
| 0 | 1 |
| 90 | 0 |
| 180 | -1 |
| 270 | 0 |
| 360 | 1 |

New points after mapping

| $\theta$ | $y$ |
|---|---|
| 180° | 9 |
| 360° | 3 |
| 540° | -3 |
| 720° | 3 |
| 900° | 9 |

| | |
|---|---|
| DOMAIN | $\{x\mid x\in\mathbb{R}\}$ |
| RANGE | $\{y\mid -3\le y\le 9, y\in\mathbb{R}\}$ |
| AMPLITUDE | 6 |
| PERIOD | $P=\frac{360°}{\frac{1}{2}}=720°$ |
| PHASE SHIFT h | 180° right |
| VERTICAL TRANSLATION | 3 up |
| EQUATION OF SINUSOIDAL AXIS | $y=3$ |



# Solutions to the homework

① $y = 3\cos(x) - 2$

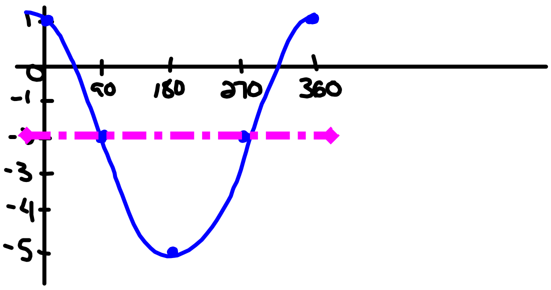
$A = 3 \quad b = 1 \quad c = 0 \quad D = -2 \quad P = 360°$

$y = \cos x$

| x | y |
|---|---|
| 0 | 1 |
| 90 | 0 |
| 180 | -1 |
| 270 | 0 |
| 360 | 1 |

→

| x | y |
|---|---|
| 0 | 1 |
| 90 | -2 |
| 180 | -5 |
| 270 | -2 |
| 360 | 1 |




② $y = -\sin(2x - \frac{\pi}{6})$

$y = -\sin[2(x - \frac{\pi}{12})]$

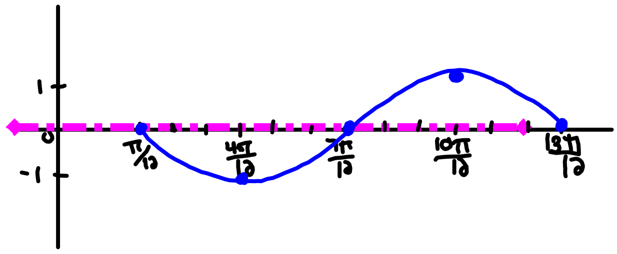
$A = 1 \quad b = 2 \quad c = \frac{\pi}{12} \quad D = 0 \quad P = \pi$

$y = \sin x$

| x | y |
|---|---|
| 0 | 0 |
| $\pi/2$ | 1 |
| $\pi$ | 0 |
| $\frac{3\pi}{2}$ | -1 |
| $2\pi$ | 0 |

→

| x | y |
|---|---|
| $\pi/12$ | 0 |
| $\pi/3$ | -1 |
| $\frac{7\pi}{12}$ | 0 |
| $5\pi/6$ | 1 |
| $13\pi/12$ | 0 |




③ $y = 4\sin(3x - 180°) + 2$

$y = 4\sin[3(x - 60°)] + 2$

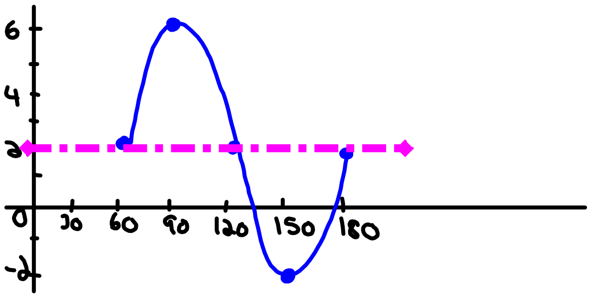
$A = 4 \quad b = 3 \quad c = 60 \quad D = 2 \quad P = 120°$

$y = \sin x$

| x | y |
|---|---|
| 0 | 0 |
| 90 | 1 |
| 180 | 0 |
| 270 | -1 |
| 360 | 0 |

→

| x | y |
|---|---|
| 60 | 2 |
| 90 | 6 |
| 120 | 2 |
| 150 | -2 |
| 180 | 2 |



⑤ $2y+3=-4\sin\left(4x-\frac{\pi}{3}\right)-3$

$2y=-4\sin\left[4\left(x-\frac{\pi}{12}\right)\right]-6$

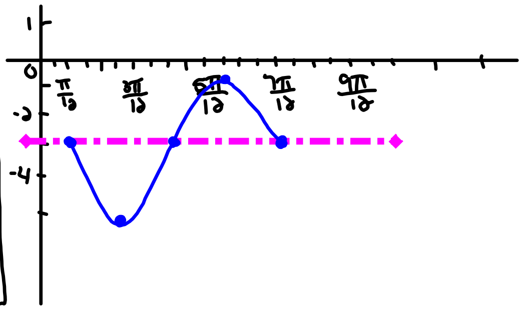
$y=-2\sin\left[4\left(x-\frac{\pi}{12}\right)\right]-3$

$A=2$ $b=4$ $C=\frac{\pi}{12}$ $D=-3$ $P=\frac{\pi}{2}$

$y=\sin x$

| x | y |
|---|---|
| 0 | 0 |
| $\pi/2$ | 1 |
| $\pi$ | 0 |
| $3\pi/2$ | -1 |
| $2\pi$ | 0 |

| x | y |
|---|---|
| $\pi/12$ | -3 |
| $\frac{5\pi}{24}$ | -5 |
| $\pi/3$ | -3 |
| $11\pi/24$ | -1 |
| $7\pi/12$ | -3 |




⑥ $\frac{y-1}{\cancel{2}}=2\cos\left(\theta+45^\circ\right)+0$

$y-1=4\cos\left(\theta+45^\circ\right)+0 \; +1$

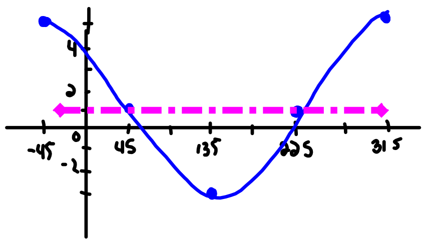
$\boxed{y=4\cos\left(\theta+45^\circ\right)+1}$

$A=4$ $b=1$ $C=-45$ $D=1$ $P=360$

$y=\cos x$

| x | y |
|---|---|
| 0 | 1 |
| 90 | 0 |
| 180 | -1 |
| 270 | 0 |
| 360 | 1 |

| x | y |
|---|---|
| -45 | 5 |
| 45 | 1 |
| 135 | -3 |
| 225 | 1 |
| 315 | 5 |




① $\frac{1}{2}y-1=\sin\left[2\left(x+\frac{\pi}{4}\right)\right]$

$\frac{1}{2}y=\sin\left[2\left(x+\frac{\pi}{4}\right)\right]+1$

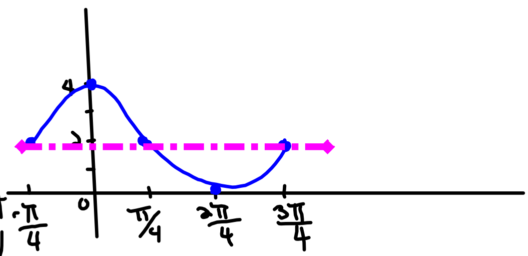
$y=2\sin\left[2\left(x+\frac{\pi}{4}\right)\right]+2$

$A=2$ $b=2$ $C=-\frac{\pi}{4}$ $D=2$ $P=\pi$

$y=\sin x$

| x | y |
|---|---|
| 0 | 0 |
| $\pi/2$ | 1 |
| $\pi$ | 0 |
| $3\pi/2$ | -1 |
| $2\pi$ | 0 |

| x | y |
|---|---|
| $-\pi/4$ | 2 |
| 0 | 4 |
| $\pi/4$ | 2 |
| $\pi/2$ | 0 |
| $\frac{3\pi}{4}$ | 2 |



8 $y = -4\cos(3x+90°)-2$

$y = -4\cos[3(x+30)]-2$

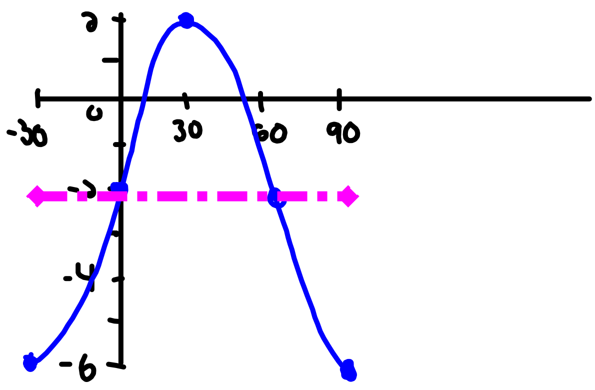
$A=4 \quad b=3 \quad c=-30 \quad D=-2 \quad P=120$

$y = \cos x$

| x | y |
|---|---|
| 0 | 1 |
| 90 | 0 |
| 180 | -1 |
| 270 | 0 |
| 360 | 1 |

$\longrightarrow$

| x | y |
|---|---|
| -30 | -6 |
| 0 | -2 |
| 30 | 2 |
| 60 | -2 |
| 90 | -6 |



# Extra Practice...

## Worksheet # 1 - 8

*Worksheet - Sketching Sinusoidal Relations*

Attachments

- worksheet-sketching in radian measure.doc
- Worksheet - Finding the Equation.doc
- Worksheet - Sketching Trigonometric Functions.doc
- Worksheet Solns - Sketching Sinusoidal Relations.doc
- Worksheet - Sketching Sinusoidal relations (sept06).pdf
- Bonus Soln - Fox Population.doc
- Worksheet Solns - Applications of Sinusoidal Relations.doc
- Review - Practice Test for Sinusoidal Functions.doc
- Review - Trigonometric Functions(3)(4).doc