Warm Up

Differentiate the following:

$$f(x) = \frac{(-2x)\left(\tan^{-1}\sqrt{x}\right)}{\cos^{-1}\left(\sec x^3\right)}$$

$$f'(x) = \frac{\overbrace{\left[-2\tan^{-1}\sqrt{x} - 2x\left(\frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}}\right)\right]}^{f'(x)}\overbrace{\left[\cos^{-1}\left(\sec x^3\right)\right]}^{g(x)} + \underbrace{\left[2x\tan^{-1}\sqrt{x}\right]}_{f(x)}\underbrace{\left[\frac{-1}{\sqrt{1-\sec^2 x^3}}\cdot \sec x^3 \tan x^3 \cdot 3x^2\right]}_{g'(x)}}{\underbrace{\left[\cos^{-1}\left(\sec x^3\right)\right]^2}_{g(x)}}$$

Questions from Homework

① e) $y = \cos^{-1}\left(\frac{x^3}{2}\right) = \cos^{-1}\left(\frac{1}{2}x^3\right)$

$$y' = \frac{-1}{\sqrt{1-\left(\frac{x^3}{2}\right)^2}} \cdot \frac{3}{2}x^2$$

$$y' = \frac{-3x^2}{2\sqrt{1-\frac{x^6}{4}}}$$

$$y' = \frac{-3x^2}{\sqrt{\underline{\underline{4}}\left(1-\frac{x^6}{4}\right)}} = \frac{-3x^2}{\sqrt{4-x^6}}$$

Questions from Homework

② $f(x) = x\tan^{-1}x$ where $x = 1$

(i) Differentiate:

$f(x) = x\tan^{-1}x$

$f'(x) = 1\tan^{-1}x + x\left(\frac{1}{1+x^2}\cdot 1\right)$

$f'(x) = \tan^{-1}x + \frac{x}{1+x^2}$

(ii) Solve $f'(1)$

what angle has a tan value of 1

$f'(1) = \boxed{\tan^{-1}(1)} + \frac{(1)}{1+(1)^2}$

(radians)

$f'(1) = \frac{\pi}{4} + \frac{1}{2}$

$\boxed{f'(1) = \frac{\pi + 2}{4}}$

⑥ If $f(x) = (x-3)\sqrt{6x-x^2} + 9\sin^{-1}\boxed{\left(\frac{x-3}{3}\right)}$ Find $f'(3)$

$\to \frac{x}{3} - \frac{3}{3}$

$\frac{1}{3}x - 1$

$f'(x) = 1\sqrt{6x-x^2} + (x-3)\frac{1}{\cancel{2}}(6x-x^2)^{-1/2}\overset{3-x}{\cancel{(6-2x)}} + \overset{3}{\cancel{9}}\left(\frac{1}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}}\cdot\frac{1}{\cancel{3}}\right)$

$f'(x) = \sqrt{6x-x^2} + \frac{(x-3)(3-x)}{\sqrt{6x-x^2}} + \frac{3}{\sqrt{1-\frac{(x-3)^2}{9}}}$

$f'(3) = \sqrt{18-9} + \frac{(0)(0)}{\sqrt{18-9}} + \frac{3}{\sqrt{1-\frac{0}{9}}}$

$f'(3) = 3 + 0 + 3 = \boxed{6}$

Questions from Homework

④ $f(x) = (3\tan^{-1}x)^4$

$$f'(x) = 4(3\tan^{-1}x)^3\left[3\left(\frac{1}{1+x^2}\cdot 1\right) + \cancel{(0)\tan^{-1}x}\right]$$

$$f'(x) = 4(3\tan^{-1}x)^3\left[\frac{3}{1+x^2}\right]$$

$$f'(x) = \frac{12(3\tan^{-1}x)^3}{1+x^2}$$

$$f'(\sqrt{3}) = \frac{12(3\tan^{-1}\sqrt{3})^3}{1+(\sqrt{3})^2}$$

what angle has a tangent value equal to $\sqrt{3}$

$$= \frac{12(3(\pi/3))^3}{1+3}$$

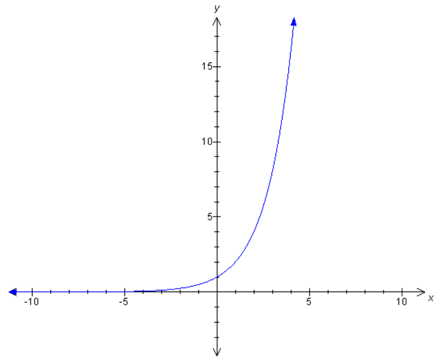
$$= \frac{12\pi^3}{4}$$

$$\boxed{= 3\pi^3}$$

Differentiating Exponential Functions

What is an exponential function?

$y = 2^x$ Base must be a Real number



When you do not have a rule to differentiate resort to the definition...

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Let's try and differentiate $y = a^x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h}$$

$$= \lim_{h\to 0}\frac{a^x a^h - a^x}{h} = \lim_{h\to 0}\frac{a^x(a^h-1)}{h}$$

This factor does not depend on h, therefore we can move to the front of the limit

Thus we now have...

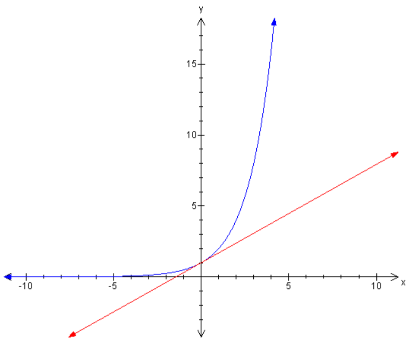
$$f'(x) = a^x \lim_{h\to 0}\frac{a^h-1}{h}$$

What would be the value of $f(0)$? $a^0 = 1$

$$\lim_{h\to 0}\frac{a^h-1}{h} = f'(0)$$

What would this represent in terms of slope??

Draw your tangent to the curve at $x = 0$



We have determined that $f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$

Same thing!

and that $\lim_{h \to 0} \frac{a^h - 1}{h} = f'(0)$

Therefore given $f(x) = a^x$, then $f'(x) = a^x f'(0)$

Here are a couple of numerical examples...

- $a = 2$; here apparently $f'(0) \approx 0.69$
- $a = 3$; here apparently $f'(0) \approx 1.10$

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

There must then be some number between 2 and 3 such that

$$\lim_{h \to 0} \frac{a^h - 1}{h} = 1$$

This number turns out to be "*e*"...Euler's Number

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e^(1)
      2.718281828
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This leads to the following definition...

Definition of the Number e

e is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

What does this mean geometrically?

- Geometrically, this means that
  - of all the exponential functions $y = a^x$,
  - the function $f(x) = e^x$ is the one whose tangent at $(0, 1)$ has a slope $f'(0)$ that is exactly 1 .

$y = 1$

$y' = 1$

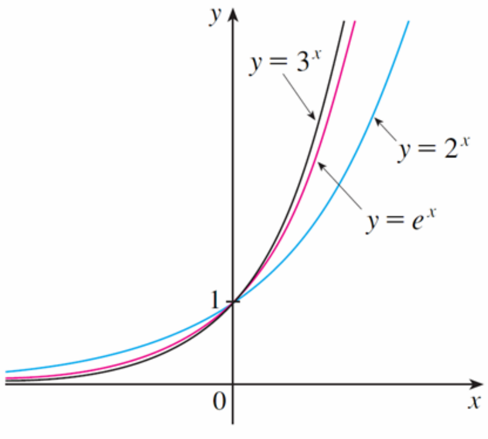

FIGURE 6

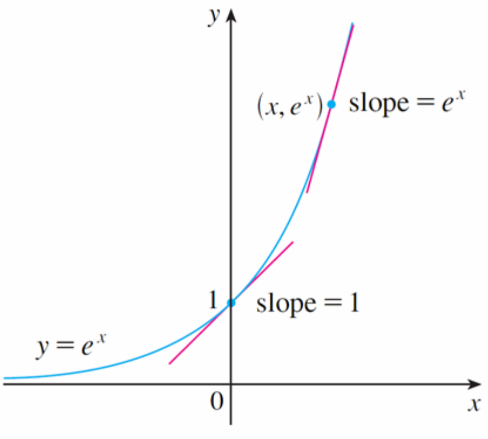

FIGURE 7

This leads to the following differentiation formula...

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

This is the ONLY function that is its own derivative

$$f(x) = e^x$$

$$\boxed{f'(x) = e^x}$$

In General...

$$\frac{d(e^u)}{dx} = e^u \bullet du$$

Differentiating Exponential Functions

$y = e^{3x^7}$ $u = 3x^7$ $du = 21x^6$

$y' = e^{3x^7} \cdot 21x^6$

$y' = 21x^6 e^{3x^7}$

$y = e^{\sin x}$ $u = \sin x$ $du = \cos x$

$y' = e^{\sin x} \cdot \cos x$

$y' = \cos x\, e^{\sin x}$

$y = (x^2)(e^x)$

$y' = 2x(e^x) + x^2 e^x$

$y' = 2xe^x + x^2e^x$

$y' = xe^x(2+x)$

$y = e^{\cot x^3}$

$y' = e^{\cot x^3} \cdot (-\csc^2(x^3) \cdot 3x^2)$

$y' = -3x^2 \csc^2(x^3)\, e^{\cot x^3}$

Practice Exercises

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#4, 5, 6, 8, 9, 10,

Bonus:

Give that $y = \cos^{-1}(\cos^{-1} x)$, prove that $\frac{dy}{dx} = \frac{1}{\sin y\sqrt{1-x^2}}$