

Understanding Logarithms

Focus on...

- demonstrating that a logarithmic function is the inverse of an exponential function
- sketching the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- determining the characteristics of the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- explaining the relationship between logarithms and exponents
- expressing a logarithmic function as an exponential function and vice versa
- evaluating logarithms using a variety of methods

General Properties of Logarithms:

If $C > 0$ and $C \neq 1$, then...

$$(i) \log_c 1 = 0$$

$$(ii) \log_c c^x = x$$

$$(iii) c^{\log_c x} = x$$

Did You Know?

The input value for a logarithm is called an argument. For example, in the expression $\log_6 1$, the argument is 1.

$$(i) \log_5 1 = 0$$

$$(ii) \log_2 2^3 = 3$$

$$(iii) 7^{\log_7 49} = 49$$

$$5^{\log_5 10} = 10$$

Product Law of Logarithms

The logarithm of a product of numbers can be expressed as the sum of the logarithms of the numbers.

$$\log_c MN = \log_c M + \log_c N$$

Proof

Let $\log_c M = x$ and $\log_c N = y$, where M , N , and c are positive real numbers with $c \neq 1$.

Write the equations in exponential form as $M = c^x$ and $N = c^y$:

$$MN = (c^x)(c^y)$$

$$MN = c^{x+y}$$

$$\log_c MN = x + y$$

$$\log_c MN = \log_c M + \log_c N$$

Apply the product law of powers.

Write in logarithmic form.

Substitute for x and y .

Quotient Law of Logarithms

The logarithm of a quotient of numbers can be expressed as the difference of the logarithms of the dividend and the divisor.

$$\log_c \frac{M}{N} = \log_c M - \log_c N$$

Proof

Let $\log_c M = x$ and $\log_c N = y$, where M , N , and c are positive real numbers with $c \neq 1$.

Write the equations in exponential form as $M = c^x$ and $N = c^y$:

$$\frac{M}{N} = \frac{c^x}{c^y}$$

$$\frac{M}{N} = c^{x-y}$$

Apply the quotient law of powers.

$$\log_c \frac{M}{N} = x - y$$

Write in logarithmic form.

$$\log_c \frac{M}{N} = \log_c M - \log_c N$$

Substitute for x and y .

Power Law of Logarithms

The logarithm of a power of a number can be expressed as the exponent times the logarithm of the number.

$$\log_c M^P = P \log_c M$$

How could you prove the quotient law using the product law and the power law?

Proof

Let $\log_c M = x$, where M and c are positive real numbers with $c \neq 1$.

Write the equation in exponential form as $M = c^x$.

Let P be a real number.

$$M = c^x$$

$$M^P = (c^x)^P$$

$$M^P = c^{xP}$$

Simplify the exponents.

$$\log_c M^P = xP$$

Write in logarithmic form.

$$\log_c M^P = (\log_c M)P$$

Substitute for x .

$$\log_c M^P = P \log_c M$$

The laws of logarithms can be applied to logarithmic functions, expressions, and equations.

Product Law of Logarithms

The logarithm of a product of numbers can be expressed as the sum of the logarithms of the numbers.

$$\log_c MN = \log_c M + \log_c N$$

$$\begin{aligned} \text{Ex: } \log_2 + \log 50 &= \log(50 \cdot 2) \\ &= \log 100 \\ &= 2 \end{aligned}$$

Quotient Law of Logarithms

The logarithm of a quotient of numbers can be expressed as the difference of the logarithms of the dividend and the divisor.

$$\log_c \frac{M}{N} = \log_c M - \log_c N$$

$$\begin{aligned} \text{Ex: } \log_2 64 - \log_2 16 &= \log_2 \left(\frac{64}{16} \right) \\ &= \log_2 4 \\ &= 2 \end{aligned}$$

Power Law of Logarithms

The logarithm of a power of a number can be expressed as the exponent times the logarithm of the number.

$$\log_c M^P = P \log_c M$$

How could you prove the quotient law using the product law and the power law?

$$\begin{aligned} \text{Ex: } \log_2 \sqrt{8} &= \log_2 8^{1/2} \\ &= \frac{1}{2} \log_2 8 \\ &= \frac{1}{2} (3) \\ &= \frac{3}{2} \text{ or } 1.5 \end{aligned}$$

Questions from Homework

Exercise 2

$$\textcircled{1} \text{ h) } \log_r v = w \text{ (log form)}$$

$$\boxed{r^w = v} \text{ (exp form)}$$

$$\textcircled{2} \text{ h) } 10^m = n \text{ (exp form)}$$

$$\log_{10} n = m$$

$$\boxed{\log n = m} \text{ (log form)}$$

$$\textcircled{3} \text{ b) } \log_8 8^{17} = 17$$

$$\text{h) } \log_4 8 = 1.5 \rightarrow \text{Let } x = \log_4 8$$

$$\frac{\log 8}{\log 4} = 1.5$$

$$4^x = 8$$

$$(2^2)^x = 2^3$$

$$2^{2x} = 2^3$$

$$2x = 3$$

$$x = \frac{3}{2} = 1.5$$

$$\textcircled{4} \text{ c) } \log_{10} (3x+5) = 2 \text{ (log form)}$$

$$10^2 = 3x+5 \text{ (exp form)}$$

$$100 = 3x+5$$

$$\frac{95}{3} = \frac{3x}{3}$$

$$\boxed{\frac{95}{3} = x}$$

$$\textcircled{4} \text{ g) } \log_3 (\log_3 x) = 4$$

↑ ans
↑ exp

Base

$$2^4 = \log_3 x$$

$$16 = \log_3 x$$

↑ exp
↑ Base
↑ ans

$$3^{16} = x$$

$$\boxed{43046721 = x}$$

$$\textcircled{4} \text{ e) } 2^{1-x} = 3 \text{ (exp form)}$$

$$\log_2 3 = 1-x \text{ (log form)}$$

$$x = 1 - \log_2 3$$

Questions from Homework

Exercise 2

$$\textcircled{3} \quad \text{b) } \log_3 3^5 = 5 \quad \text{g) } \log_3 \left(\frac{1}{27}\right) = -3$$

$$\frac{\log 3^5}{\log 3} = 5 \quad \frac{\log \left(\frac{1}{27}\right)}{\log 3} = -3$$

$$\text{j) } \log_9 \sqrt{3} = 0.25$$

$$\frac{\log(\sqrt{3})}{\log 9} = 0.25 \text{ or } \frac{1}{4}$$

$$\textcircled{4} \quad \text{f) } 3^{2x-1} = 5 \quad (\text{exp form})$$

\uparrow base
 \uparrow ans

$$\log_3(5) = 2x - 1$$

$$\frac{\log_3(5) + 1}{2} = \frac{2x}{2}$$

$$\frac{\log_3(5) + 1}{2} = x$$

$$\frac{1}{2}(\log_3(5) + 1) = x$$

$$\text{h) } 10^{5^x} = 3 \quad (\text{Exp. form})$$

\uparrow Base
 \uparrow ans

$$\log_{10}(3) = 5^x \quad (\text{Exp form})$$

\uparrow ans
 \uparrow Base

$$\log_5(\log_{10} 3) = x$$

Example 1

Use the Laws of Logarithms to Expand Expressions

Write each expression in terms of individual logarithms of x , y , and z .

a) $\log_5 \frac{xy}{z}$

b) $\log_7 \sqrt[3]{x}$

c) $\log_6 \frac{1}{x^2}$

d) $\log \frac{x^3}{y\sqrt{z}}$

$$\text{a) } \log_5 \frac{xy}{z} = \boxed{\log_5 x + \log_5 y - \log_5 z}$$

$$\text{b) } \log_7 \sqrt[3]{x} = \log_7 x^{\frac{1}{3}} = \boxed{\frac{1}{3} \log_7 x}$$

$$\text{c) } \log_6 \frac{1}{x^2} = \log_6 1 - \log_6 x^2$$

$$= 0 - 2 \log_6 x$$

$$= \boxed{-2 \log_6 x}$$

$$\text{d) } \log \frac{x^3}{y\sqrt{z}} = \log x^3 - (\log y + \log \sqrt{z})$$

$$= \log x^3 - \log y - \log z^{\frac{1}{2}}$$

$$= \boxed{3 \log x - \log y - \frac{1}{2} \log z}$$

Example 2

Use the Laws of Logarithms to Evaluate Expressions

Use the laws of logarithms to simplify and evaluate each expression.

a) $\log_6 8 + \log_6 9 - \log_6 2$

b) $\log_7 7\sqrt{7}$

c) $2 \log_2 12 - (\log_2 6 + \frac{1}{3} \log_2 27)$

a) $\log_6 8 + \log_6 9 - \log_6 2$

$$= \log_6 \left(\frac{8 \cdot 9}{2} \right)$$

$$= \log_6 (36)$$

$$= 2$$

b) $\log_7 7\sqrt{7}$

$$= \log_7 7 + \log_7 \sqrt{7}$$

$$= \log_7 7 + \log_7 7^{1/2}$$

$$= 1 + \frac{1}{2}$$

$$= \frac{2}{2} + \frac{1}{2} = \left(\frac{3}{2} \right)$$

c) $2 \log_2 12 - (\log_2 6 + \frac{1}{3} \log_2 27)$

$$\log_2 144 - (\log_2 6 + \log_2 3)$$

$$\log_2 144 - \log_2 6 - \log_2 3$$

$$\log_2 \left(\frac{144}{6 \cdot 3} \right)$$

$$\log_2 8$$

$$3$$

$$\log_7 7\sqrt{7}$$

$$\log_7 7(7)^{\frac{1}{2}}$$

$$\log_7 7^{1+\frac{1}{2}}$$

$$\log_7 7^{\frac{3}{2}}$$

$$\frac{3}{2}$$

Example 3

Use the Laws of Logarithms to Simplify Expressions

Write each expression as a single logarithm in simplest form. State the restrictions on the variable.

a) $\log_7 x^2 + \log_7 x - \frac{5 \log_7 x}{2}$

b) $\log_5 (2x - 2) - \log_5 (x^2 + 2x - 3)$

a) $\log_7 x^2 + \log_7 x - \left(\frac{5}{2}\right) \log_7 x$

$\log_7 x^2 + \log_7 x - \log_7 x^{5/2}$

$\log_7 \left(\frac{x^2 \cdot x}{x^{5/2}}\right)$

$\log_7 \left(\frac{x^3}{x^{5/2}}\right)$

$\log_7 x^{3-5/2}$

$\log_7 x^{6/2-5/2}$

$\log_7 x^{1/2}$

$\frac{1}{2} \log_7 x$

b) $\log_5 (2x-2) - \log_5 (x^2+2x-3)$

$\log_5 \left(\frac{2x-2}{x^2+2x-3}\right)$
 ← common factor
 ← trinomial

$\log_5 \left[\frac{2(x-1)}{(x-1)(x+3)}\right]$

$-1+3=2$
 $-1 \times 3 = -3$

$\log_5 \left(\frac{2}{x+3}\right)$

For the original expression to be defined, both logarithmic terms must be defined.

$2x - 2 > 0$ $x^2 + 2x - 3 > 0$
 $2x > 2$ $(x + 3)(x - 1) > 0$
 $x > 1$ and $x < -3$ or $x > 1$

What other methods could you have used to solve this quadratic inequality?

The conditions $x > 1$ and $x < -3$ or $x > 1$ are both satisfied when $x > 1$.

Hence, the variable x needs to be restricted to $x > 1$ for the original expression to be defined and then written as a single logarithm.

Therefore, $\log_5 (2x - 2) - \log_5 (x^2 + 2x - 3) = \log_5 \frac{2}{x + 3}, x > 1$.

Key Ideas

- Let P be any real number, and M , N , and c be positive real numbers with $c \neq 1$. Then, the following laws of logarithms are valid.

Name	Law	Description
Product	$\log_c MN = \log_c M + \log_c N$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
Quotient	$\log_c \frac{M}{N} = \log_c M - \log_c N$	The logarithm of a quotient of numbers is the difference of the logarithms of the dividend and divisor.
Power	$\log_c M^P = P \log_c M$	The logarithm of a power of a number is the exponent times the logarithm of the number.

- Many quantities in science are measured using a logarithmic scale. Two commonly used logarithmic scales are the decibel scale and the pH scale.

Homework

Finish Exercise 3

Questions from Homework

Exercise 3

$$\textcircled{3} \text{ e) } \frac{1}{2} [\log_5 x + \textcircled{2} \log_5 y - \textcircled{3} \log_5 z]$$

$$\frac{1}{2} [\log_5 x + \log_5 y^2 - \log_5 z^3]$$

$$\textcircled{\frac{1}{2}} \log_5 \left(\frac{xy^2}{z^3} \right)$$

$$\log_5 \left(\frac{xy^2}{z^3} \right)^{\frac{1}{2}}$$

$$\log_5 \sqrt{\frac{xy^2}{z^3}}$$

$$\log_5 y \sqrt{\frac{x}{z^3}}$$

Do I really understand??...

a) Express the following as a single logarithm... $2\log_2 3^2 + \log_2 6 - 3\log_2 3$

b) Evaluate the following... $\log_2 (32)^{\frac{1}{3}}$

c) Express the following as a single logarithm... $\frac{1}{2}[(\log_5 a + 2\log_5 b) - 3\log_5 c]$

d) Express as a single logarithm in simplest form...

$$\frac{3}{4} \left[12(\log_8 x^2 - 2\log_8 x) + 8\log_8 \sqrt{x} - 4\log_8 \frac{1}{x^7} \right]$$