

Exam Outline:

- Limit definition of the derivative
- Derivatives (Do not Simplify / Simplify)
- Limits
- Curve Sketch
- Related Rate Problem
- Optimization Problem
- Riemann Sum (#6 Integration)
- Integrals (Indefinite / Definite)
- Area Between Curves, (#9 Integration)

$$\textcircled{1} \text{ i) } y = x^x$$

$$\ln y = \ln x^x$$

$$\ln y = (x)(\ln x)$$

← product rule

$$y \cdot \frac{y'}{y} = \left[(1)(\ln x) + x\left(\frac{1}{x}\right) \right] \cdot y$$

$$y' = (\ln x + 1) \underline{x^x} = x^x (\ln x + 1)$$

$$\textcircled{3} \quad y = \frac{(3x+4)^8 \sqrt{2x^3-5}}{x^{14}} = \frac{(3x+4)^8 (2x^3-5)^{1/2}}{x^{14}}$$

$$\ln y = \ln \left(\frac{(3x+4)^8 (2x^3-5)^{1/2}}{x^{14}} \right)$$

$$\ln y = \ln(3x+4)^8 + \ln(2x^3-5)^{1/2} - \ln x^{14}$$

$$\ln y = 8 \ln(3x+4) + \frac{1}{2} \ln(2x^3-5) - 14 \ln x$$

$$\frac{y'}{y} = 8 \left(\frac{3}{3x+4} \right) + \frac{1}{2} \left(\frac{6x^2}{2x^3-5} \right) - 14 \left(\frac{1}{x} \right)$$

$$y \cdot \frac{y'}{y} = \left[\frac{24}{3x+4} + \frac{3x^2}{2x^3-5} - \frac{14}{x} \right] y$$

$$y' = \left[\frac{24}{3x+4} + \frac{3x^2}{2x^3-5} - \frac{14}{x} \right] \left[\frac{(3x+4)^8 \sqrt{2x^3-5}}{x^{14}} \right]$$

$$\textcircled{a} \quad x^2 + 3xy = y^2 + 2x + 1$$

$$\text{a) } \begin{array}{l} x = -2 \\ y = 1 \end{array} \quad \begin{array}{l} (-2)^2 + 3(-2)(1) = (1)^2 + 2(-2) + 1 \\ 4 - 6 \quad \quad \quad | \quad 1 - 4 + 1 \\ -2 \quad \quad \quad \quad | \quad -2 \quad \checkmark \end{array}$$

$$\text{b) } x^2 + (3xy) = y^2 + 2x + 1$$

$$2x + 3y + 3x \frac{dy}{dx} = 2y \frac{dy}{dx} + 2$$

$$2x + 3y - 2 = 2y \frac{dy}{dx} - 3x \frac{dy}{dx}$$

$$2x + 3y - 2 = \frac{dy}{dx} (2y - 3x)$$

$$\boxed{\frac{2x + 3y - 2}{2y - 3x} = \frac{dy}{dx}} \quad \leftarrow m$$

c) ① Find m:

$$m = \frac{dy}{dx} = \frac{2(-2) + 3(1) - 2}{2(1) - 3(-2)} = \frac{-3}{8}$$

$$\textcircled{2} \quad y - y_1 = m(x - x_1)$$

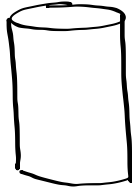
$$y - 1 = \frac{-3}{8}(x + 2)$$

$$\textcircled{8} \quad y - 1 = \frac{-3x}{8} - \frac{6}{8}$$

$$8y - 8 = -3x - 6$$

$$\boxed{3x + 8y - 2 = 0}$$

10



$$V = \pi r^2 h$$

$$A = 2\pi r^2 + 2\pi r h$$

$$2000 = \pi r^2 h$$

$$A = 2\pi r^2 + 2\pi r \left(\frac{2000}{\pi r^2} \right)$$

$$\frac{2000}{\pi r^2} = h$$

$$A = 2\pi r^2 + \frac{4000\pi r}{\pi r^2}$$

$$A = 2\pi r^2 + 4000 r^{-1}$$

$$A' = 4\pi r - 4000 r^{-2}$$

$$A' = \frac{4\pi r}{1} - \frac{4000}{r^2}$$

$$A' = \frac{4\pi r^3 - 4000}{r^2}$$

$$A' = \frac{4\pi r^3 - 4000}{r^2}$$

$$0 = 4\pi r - \frac{4000}{r^2}$$

$$\frac{4000}{r^2} = 4\pi r$$

$$4\pi r^3 = 4000$$

$$r^3 = \frac{1000}{\pi}$$

$$r = \frac{10}{\sqrt[3]{\pi}} \approx 9,55 \text{ cm}$$

Critical Values

$$4\pi r^3 - 4000 = 0$$

$$r^2 = 0$$

$$\frac{4\pi r^3}{4\pi} = \frac{4000}{4\pi}$$

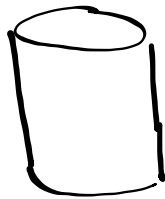
$$r = 0$$

$$r^3 = \frac{1000}{\pi}$$

$$r = \frac{\sqrt[3]{1000}}{\sqrt[3]{\pi}}$$

$$r = \frac{10}{\sqrt[3]{\pi}} \text{ min}$$

②



$$A = 2\pi r^2 + 2\pi r h$$

$$2400 = 2\pi r^2 + 2\pi r h$$

$$2400 - 2\pi r^2 = 2\pi r h$$

$$\frac{2400 - 2\pi r^2}{2\pi r} = h$$

$$\boxed{\frac{1200 - \pi r^2}{\pi r}} = h$$

$$V = \pi r^2 h$$

$$V = \pi r^2 \left(\frac{1200 - \pi r^2}{\pi r} \right)$$

$$V = 1200r - \pi r^3$$



$$\textcircled{1a} \text{ b) } f(x) = x^2 \csc^2(3x^2) = \underbrace{x^2}_{f(x)} \underbrace{(\csc(3x^2))^2}_{g(x)}$$

$$f'(x) = 2x \csc^2(3x^2) + 2x^2 \csc(3x^2) (-\csc(3x^2) \cot(3x^2)) (6x)$$

$$f'(x) = 2x \csc^2(3x^2) - 12x^3 \csc^2(3x^2) \cot(3x^2)$$

$$f'(x) = 2x \csc^2(3x^2) [1 - 6x^2 \cot(3x^2)]$$

1. [4 points each] Find $\frac{dy}{dx}$ for each of the following. Do **NOT** simplify your answers.

(a) $y = \frac{6 - x^2 + \sqrt{x}}{x^3 + 4}$

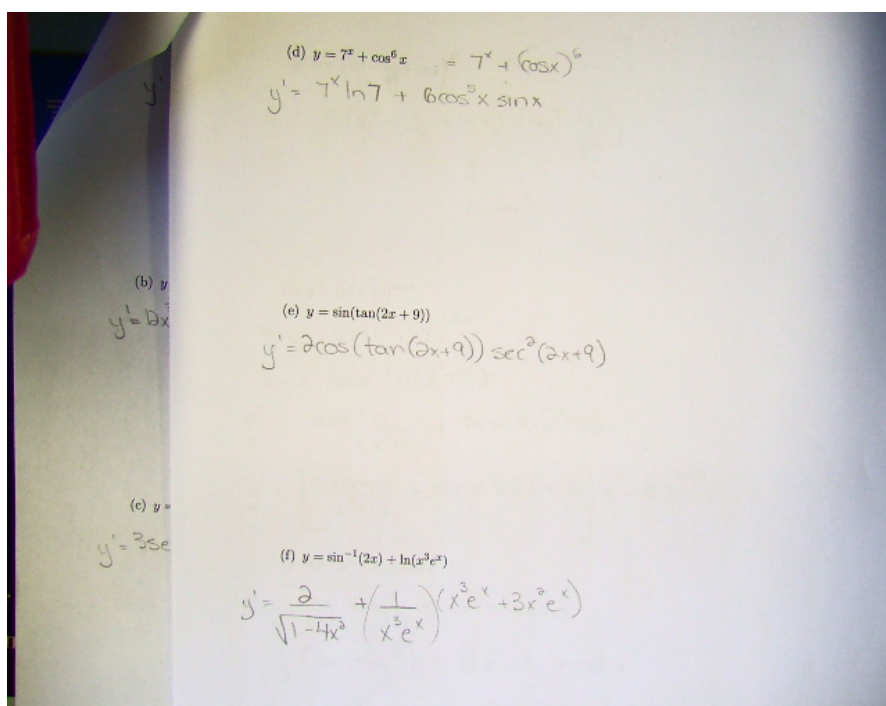
$$y' = \frac{(x^3+4)(-2x + \frac{1}{2}x^{-1/2}) - (6-x^2+\sqrt{x})(3x^2)}{(x^3+4)^2}$$

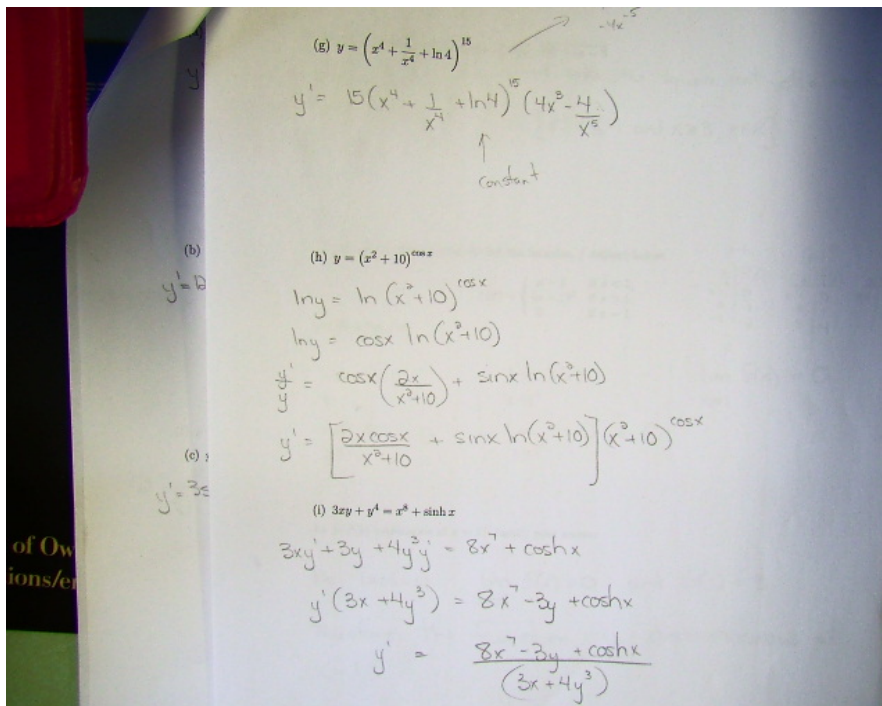
(b) $y = e^x \sin(3x^4 + 5)$

$$y' = 3x^3 e^x \cos(3x^4 + 5) + e^x \sin(3x^4 + 5)$$

(c) $y = \sec(3x) + \sqrt{\pi x - 5}$

$$y' = 3 \sec(3x) \tan(3x) + \frac{1}{2}(\pi x - 5)^{-1/2} (\pi)$$





2. [4 points] Find the domain of $f(x) = \sqrt{x^2 - 2x - 3}$

$x^2 - 2x - 3 \geq 0$ ← cannot take the square root of a negative

$y = (x-3)(x+1)$

D: $\{x \mid x \leq -1 \text{ and } x \geq 3, x \in \mathbb{R}\}$

3. [3 points] Answer (a) and (b) for the function f defined below.

$$f(x) = \begin{cases} x-1 & \text{if } x < 1; \\ (x-1)^2 & \text{if } x > 1; \\ 3 & \text{if } x = 1. \end{cases}$$

(a) Find $\lim_{x \rightarrow 1} f(x)$.

$\lim_{x \rightarrow 1^-} f(x)$ $\lim_{x \rightarrow 1^-} (x-1) = 0$	$\lim_{x \rightarrow 1^+} f(x)$ $\lim_{x \rightarrow 1^+} (x-1)^2 = 0$	$\lim_{x \rightarrow 1} f(x) = 0$
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(b) Is $f(x)$ continuous at $x = 1$? Justify your answer.

No because $\lim_{x \rightarrow 1} f(x) = 0$ and $f(1) = 3$

(a) $\lim_{x \rightarrow 0} \frac{e^x - 1}{\cos x}$
 $\lim_{x \rightarrow 0} \frac{e^x}{-\sin x} = \frac{1}{0} = \text{DNE}$

(b) $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 - x - 2}$
 $\lim_{x \rightarrow 2} \frac{2x}{2x - 1} = \frac{4}{3}$

(c) $\lim_{x \rightarrow \infty} \frac{1 - 3x^2}{x^2 - 5x + 7}$
 $\lim_{x \rightarrow \infty} \frac{\frac{1}{x^2} - 3}{1 - \frac{5}{x} + \frac{7}{x^2}} = \frac{-3}{1} = -3$

(d) $\lim_{x \rightarrow 1} \frac{x^2 + 1}{x(x-1)}$
 $\lim_{x \rightarrow 1} \frac{x^2 + 1}{x(x-1)} = -\infty$
 ↑
 denominator is a very small negative
 $x=1$

Use the limit definition of the derivative to find $f'(x)$, given $f(x) = x^2 - 3x + 5$.

$$f(x+h) = x^2 + 2xh + h^2 - 3x - 3h + 5$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 3x - 3h + 5 - x^2 + 3x - 5}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{h(2x + h - 3)}{h} = \boxed{2x - 3}$$

6. Answer (a)-(c) with regard to the function $f(x) = \sqrt{x-4}$. $\rightarrow D: \{x \geq 4, x \in \mathbb{R}\}$
 $R: \{y \geq 0, y \in \mathbb{R}\}$

(a) [3 points] Find the inverse function $f^{-1}(x)$.

$y = \sqrt{x-4}$
 $x = \sqrt{y-4}$
 $x^2 = y-4$
 $x^2 + 4 = y$
 $f^{-1}(x) = x^2 + 4$

x	f(x)
4	0
5	1
8	2
13	3
20	4

 \rightarrow

x	f^{-1}(x)
0	4
1	5
2	8
3	13
4	20

(b) [2 points] Find the domain and range of $f^{-1}(x)$ (the inverse function you found in part (a)).

Domain $\{x \mid x \geq 0, x \in \mathbb{R}\}$

Range $\{y \mid y \geq 4, y \in \mathbb{R}\}$

(c) [2 points] Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate system (axes).

7. [3 points] Suppose that $h(x) = f(x)g(x)$ where $f(2) = 3$, $g(2) = 5$, $f'(2) = -2$, and $g'(2) = 4$. Find $h'(2)$.

$$h'(2) = f(2)g'(2) + f'(2)g(2)$$

$$= (3)(4) + (-2)(5)$$

$$= 12 - 10$$

$$= 2$$

8. [5 points] Find the equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.

Point is not on curve.

$y = x^2 + x$
 $y' = 2x + 1$
 \uparrow
 m

To Find points of tangency use $y - y_1 = m(x - x_1)$
 $y - (-3) = (2x + 1)(x - 2)$
 $y + 3 = 2x^2 - 3x - 2$
 $x^2 + x + 3 = 2x^2 - 3x - 2$
 $0 = x^2 - 4x - 5$
 $0 = (x - 5)(x + 1)$
 $x = 5$ and $x = -1$

Slope @ $x = 5$
 $y' = 2(5) + 1 = 11$
 Equation
 $y + 3 = 11(x - 2)$
 $y = 11x - 22 - 3$
 $y = 11x - 25$

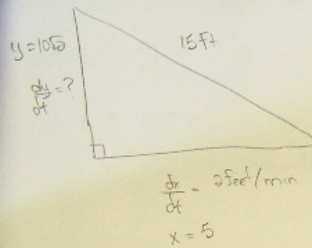
Slope @ $x = -1$
 $y' = 2(-1) + 1 = -1$
 Equation
 $y + 3 = -1(x - 2)$
 $y = -x - 1$

9. [4 points] Find the critical points of the function $f(x) = \ln(2 + \sin x)$ on the interval $[0, 2\pi]$.

$f(x) = \ln(2 + \sin x)$
 $f'(x) = \frac{1}{2 + \sin x} \cdot \cos x$
 $f'(x) = \frac{\cos x}{2 + \sin x}$

CV: $\cos x = 0$ | $2 + \sin x = 0$
 $x = \frac{\pi}{2}, \frac{3\pi}{2}$ | $\sin x = -2$
 Not possible

10. [6 points] A 15 foot ladder is leaning on a wall of a house. The bottom of the ladder is pulled away from the base of the wall at a constant rate of 2 feet per minute. At what rate is the top of the ladder sliding down the wall when the bottom of the ladder is 5 feet from the wall?



$$x^2 + y^2 = 15^2$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$2(5)(2) + 2(10\sqrt{3}) \frac{dy}{dt} = 0$$

$$20\sqrt{3} \frac{dy}{dt} = -20$$

$$\frac{dy}{dt} = \frac{-1}{\sqrt{3}} \text{ ft/min}$$

$$y = \sqrt{15^2 - 5^2}$$

$$y = \sqrt{225 - 25}$$

$$y = \sqrt{200}$$

$$y = 10\sqrt{3}$$

The top of the ladder is sliding down the wall at a rate of $\frac{1}{\sqrt{3}}$ ft/min

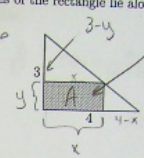
$$\text{or } \frac{\sqrt{3}}{3}$$

11. [6 points] Find the area of the largest rectangle that can be inscribed in a right triangle with legs of 3cm and 4cm if two sides of the rectangle lie along the legs.

express with a single variable

maximize Area

similar triangles:



$A = xy$

$A = x(3 - \frac{3}{4}x)$

$A = 3x - \frac{3}{4}x^2$

Differentiate

$A' = 3 - \frac{3}{2}x$

$\frac{3}{2}x = 3$

$3x = 6$

$x = 2$

$\frac{3-y}{x} = \frac{3}{4}$

$3x = 12 - 4y$

$4y = 12 - 3x$

$y = 3 - \frac{3}{4}x$

$y = 3 - \frac{3}{4}(2)$

$y = 3 - \frac{3}{2}$

$y = \frac{6-3}{2} = \frac{3}{2}$

The maximum area is 3cm^2

Dimensions that maximize area are $2\text{cm} \times 1.5\text{cm}$

$A = xy$

$A = 2(\frac{3}{2})$

$A = 3\text{cm}^2$

12. [13 points] Answer (a)-(h) with regard to the function $f(x) = \frac{x^2 - 2x + 4}{x - 2}$. The first and second derivatives of f are given below.

$$f'(x) = \frac{x(x-4)}{(x-2)^2} \quad f''(x) = \frac{8}{(x-2)^3}$$

(a) Find the intercepts, if any.

y int (x=0) $y = \frac{4}{-2} = -2$ $(0, -2)$

x int (y=0) $x^2 - 2x + 4 = 0$
 $x = \frac{2 \pm \sqrt{4 - 16}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3}$ \rightarrow Imaginary Roots
 No x intercepts

(b) Find the horizontal and vertical asymptotes, if any.

Horizontal: None $\lim_{x \rightarrow \infty} \frac{x^2 - 2x + 4}{x - 2} = \text{DNE}$
 Vertical: $x = 2$ $\lim_{x \rightarrow 2^-} \frac{x^2 - 2x + 4}{x - 2} = -\infty$
 $\lim_{x \rightarrow 2^+} \frac{x^2 - 2x + 4}{x - 2} = +\infty$
 Slant: $y = x$ $\lim_{x \rightarrow \infty} \frac{x^2 - 2x + 4}{x - 2} = \frac{x^2 - 2x + 4}{x - 2} = x - 4 + \frac{8}{x - 2}$

(c) Find the critical numbers for f .

$S(x) = \frac{x(x-4)}{(x-2)^2}$ CV: $x = 0, 2, 4$

(d) Determine the intervals where f is increasing and the intervals where f is decreasing.

Sign chart for f' :

$x < 0$	$0 < x < 2$	$2 < x < 4$	$x > 4$
+	-	-	+
$(-\infty, 0)$	$(0, 2)$	$(2, 4)$	$(4, \infty)$
+	-	-	+

f is increasing on $(-\infty, 0) \cup (4, \infty)$
 f is decreasing on $(0, 2) \cup (2, 4)$



