Making a Complete Sketch

① Plot all points: x-int, y-int, max, min, I.P.,

② Plot all asymptotes (Check behaviour near VA.)

③ Use intervals of inc/dec and concavity to connect everything

Example:

Examine the function $f(x)=3x^5-5x^3$ with respect to…

- Intercepts $f(x)$
- ~~Symmetry~~
- Asymptotes (No asymptotes for polynomial functions)
- Intervals of Increase or Decrease $f'(x)$
- Local Maximum and Minimum values $f(x)$
- $f''(x)$ ~~Concavity~~ and ~~Points of Inflection~~ $f(x)$
- Sketch the Curve

$f(x)=3x^5-5x^3$ $\quad f'(x)=15x^4-15x^2$ $\quad f''(x)=60x^3-30x$

$f(x)=x^3(3x^2-5)$ $\quad f'(x)=15x^2(x^2-1)$ $\quad f''(x)=30x(2x^2-1)$

$f'(x)=15x^2(x-1)(x+1)$

① x-int $(y=0)$

$f(x)=x^3(3x^2-5)$

$0=x^3(3x^2-5)$

$x^3=0$	$3x^2-5=0$
$x=0$	$\frac{3x^2}{3}=\frac{5}{3}$
$(0,0)$	$x^2=\frac{5}{3}$
	$x=\pm\sqrt{5/3}$
	$(1.29,0)$
	$+(-1.29,0)$

② y-int $(x=0)$

$f(x)=3x^5-5x^3$

$f(0)=3(0)^5-5(0)^3$

$f(0)=0$

$(0,0)$

③ Intervals of Inc/Dec.

$f'(x)=15x^2(x-1)(x+1)$

$0=15x^2(x-1)(x+1)$

$15x^2=0$	$x-1=0$	$x+1=0$
$x^2=0$	$x=1$	$x=-1$
$x=0$		

CV: $x=-1,0,1$

max min

$+\ \ -\ \ -\ \ +$

$(-2)\ \ -1\ \ (-\frac{1}{2})\ \ 0\ \ (\frac{1}{2})\ \ 1\ \ (2)$

Increasing on $(-\infty,-1)+(1,\infty)$

Decreasing on $(-1,0)+(0,1)$ or $(-1,1)$

④ max @ $x=-1$

$f(x)=3x^5-5x^3$

$f(-1)=3(-1)^5-5(-1)^3$

$f(-1)=-3+5$

$f(-1)=2$

$(-1,2)$

⑤ min @ $x=1$

$f(x)=3x^5-5x^3$

$f(1)=3(1)^5-5(1)^3$

$f(1)=3-5$

$f(1)=-2$

$(1,-2)$

⑥ Intervals of Concavity:

$f''(x)=30x(2x^2-1)$

$0=30x(2x^2-1)$

$30x=0$	$2x^2-1=0$
$x=0$	$2x^2=1$
	$x^2=\frac{1}{2}$
	$x=\pm\sqrt{1/2}$
	$x\approx\pm0.707$

CV: $x=-0.7,0,0.7$

I.P. I.P. I.P.

$-\ \ +\ \ -\ \ +$

$(-1)\ \ -0.7\ \ (-0.5)\ \ 0\ \ (0.5)\ \ 0.7\ \ (1)$

CD on $(-\infty,-0.7)+(0,0.7)$

CU on $(-0.7,0)+(0.7,\infty)$

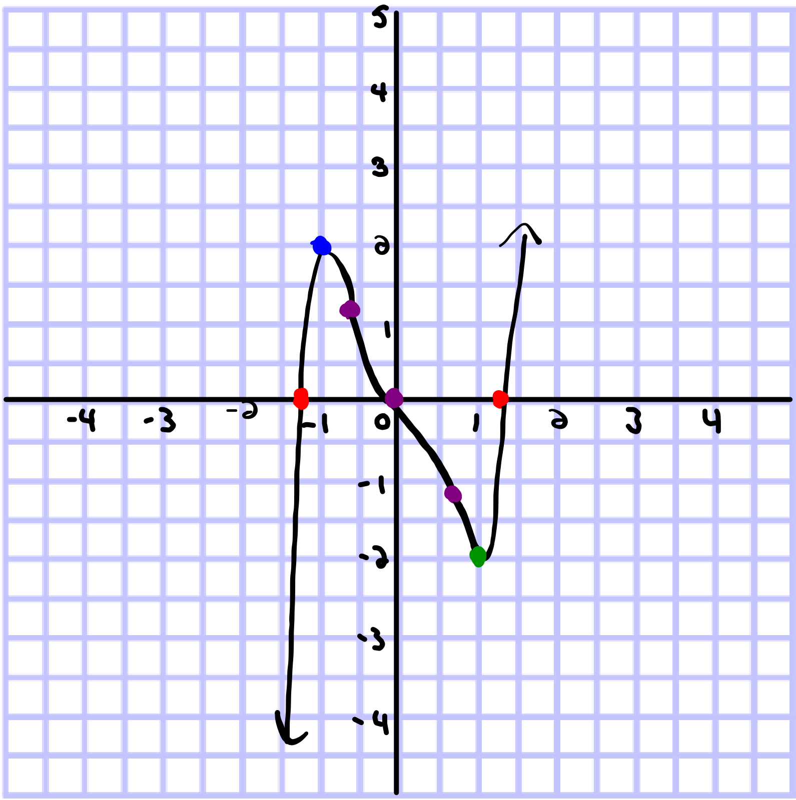
⑦ Inflection Points

$f(x)=3x^5-5x^3$

$f(-0.7)=3(-0.7)^5-5(-0.7)^3=-0.504+1.715=1.2$ $\quad(-0.7,1.2)$

$f(0)=3(0)^5-5(0)^3=0-0=0$ $\quad(0,0)$

$f(0.7)=3(0.7)^5-5(0.7)^3=0.504-1.715=-1.2$ $\quad(0.7,-1.2)$


5
4
3
2
1
-4
-3
-2
-1
0
1
2
3
4
-1
-2
-3
-4

Assignment:

$f(x) = x^2 + x^3$ $\quad f'(x) = 2x + 3x^2$ $\quad f''(x) = 2 + 6x$

$f(x) = x^2(1+x)$ $\quad f'(x) = x(2+3x)$ $\quad f''(x) = 2(1+3x)$

① x-int $(y=0)$

$f(x) = x^2(1+x)$

$0 = x^2(1+x)$

$x^2 = 0$ | $1+x=0$

$x=0$ | $x=-1$

$(0,0)$ | $(-1,0)$

② y-int $(x=0)$

$f(x) = x^2 + x^3$

$f(0) = (0)^2 + (0)^3$

$f(0) = 0 + 0$

$f(0) = 0$

$(0,0)$

③ Intervals of Inc/Dec.

$f'(x) = x(2+3x)$

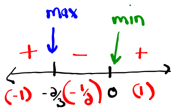
$0 = x(2+3x)$

$x = 0$ | $2+3x=0$

| $3x = -2$

| $x = -2/3$

CV: $x = -\frac{2}{3}, 0$



Increasing on $(-\infty, -2/3)$ + $(0, \infty)$

Decreasing on $(-2/3, 0)$

④ max @ $x = -\frac{2}{3}$

$f(x) = x^2 + x^3$

$f(-\frac{2}{3}) = (-\frac{2}{3})^2 + (-\frac{2}{3})^3$

$f(-\frac{2}{3}) = \frac{4}{9} - \frac{8}{27}$

$f(-\frac{2}{3}) = \frac{12}{27} - \frac{8}{27} = \frac{4}{27}$

$(-\frac{2}{3}, \frac{4}{27})$ or $(-0.\overline{6}, 0.15)$

⑤ min @ $x=0$

$f(x) = x^2 + x^3$

$f(0) = (0)^2 + (0)^3$

$f(0) = 0 + 0 = 0$

$(0,0)$

⑥ Intervals of Concavity:

$f''(x) = 2(1+3x)$

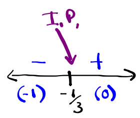
$0 = 2(1+3x)$

$2 \neq 0$ | $1+3x=0$

| $3x=-1$

| $x = -\frac{1}{3}$

CV: $x = -\frac{1}{3}$



CD on $(-\infty, -\frac{1}{3})$

CU on $(-\frac{1}{3}, \infty)$

⑦ Inflection Point: @ $x = -\frac{1}{3}$

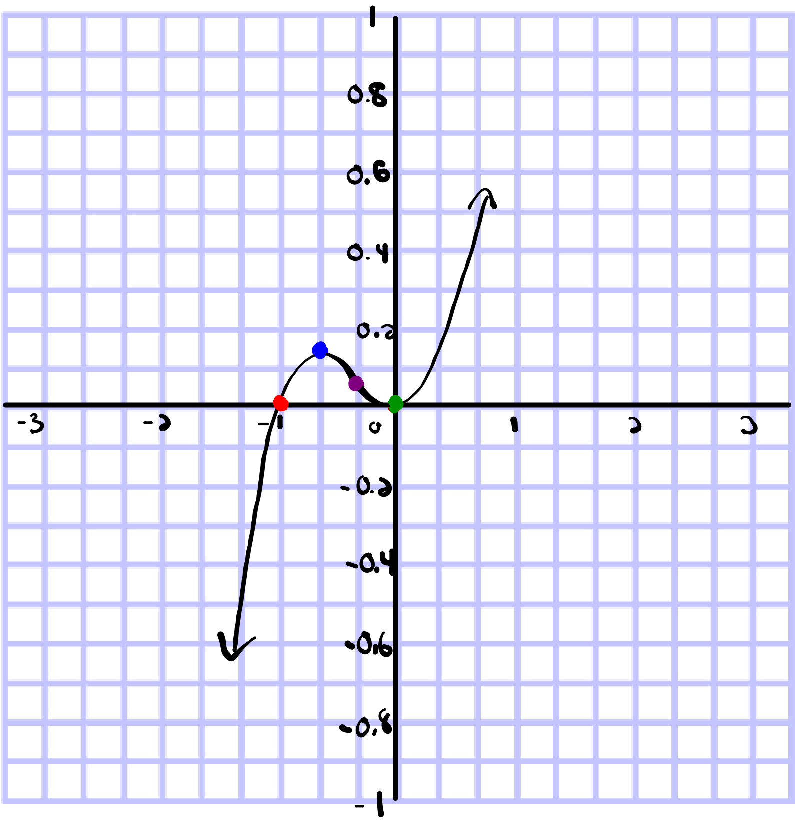
$f(x) = x^2 + x^3$

$f(-\frac{1}{3}) = (-\frac{1}{3})^2 + (-\frac{1}{3})^3$

$f(-\frac{1}{3}) = \frac{1}{9} - \frac{1}{27}$

$f(-\frac{1}{3}) = \frac{3}{27} - \frac{1}{27} = \frac{2}{27}$

$(-\frac{1}{3}, \frac{2}{27})$ or $(-0.\overline{3}, 0.07)$


1
0.8
0.6
0.4
0.2
-3
-2
-1
0
1
2
3
-0.2
-0.4
-0.6
-0.8
-1

homework

Examine the function $f(x)=\frac{x^2}{1-x^2}$ with respect to...

$f'(x)=\frac{2x}{(1-x^2)^2}$

$f''(x)=\frac{2(1+3x^2)}{(1-x^2)^3}$

- Intercepts $f(x)$
- ~~Symmetry~~
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

① x-int (y=0)

$f(x)=\frac{x^2}{1-x^2}$

$0=\frac{x^2}{1-x^2}$

$0=x^2$

$0=x$

(0,0)

② y-int (x=0)

$f(x)=\frac{x^2}{1-x^2}$

$f(0)=\frac{(0)^2}{1-(0)^2}$

$f(0)=\frac{0}{1}=0$

(0,0)

③ Vertical Asymptote: (zeroes of the denominator)

$f(x)=\frac{x^2}{1-x^2}$

VA: $1-x^2=0$

$(1-x)(1+x)=0$

$1-x=0$ | $1+x=0$

$\boxed{1=x}$ | $\boxed{x=-1}$

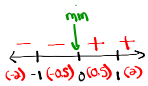
④ Horizontal Asymptote:

$\lim_{x\to\infty}\frac{x^2}{1-x^2}=\frac{1}{-1}=-1$

$\boxed{y=-1}$

⑤ Intervals of Inc/Dec.

$f'(x)=\frac{2x}{(1-x^2)^2}$



$2x=0$ | $(1-x^2)^2=0$

$x=0$ | $1-x^2=0$

| $1=x^2$

| $\pm1=x$

CV: $x=-1,0,1$

Increasing on $(0,\infty)$

Decreasing on $(-\infty,0)$

⑥ min @ x=0

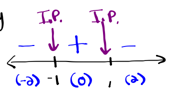
$f(x)=\frac{x^2}{1-x^2}$

$f(0)=\frac{(0)^2}{1-(0)^2}=0$

(0,0)

⑦ Intervals of Concavity

$f''(x)=\frac{2(1+3x^2)}{(1-x^2)^3}$



$2(1+3x^2)=0$ | $(1-x^2)^3=0$

$1+3x^2=0$ | $1-x^2=0$

$3x^2=-1$ | $1=x^2$

$x^2=-\frac{1}{3}$ | $\pm1=x$

Not Possible

(Numerator is always positive)

CD on $(-\infty,-1)+(1,\infty)$

CU on $(-1,1)$

⑧ Inflection Points:

when x=-1

$f(x)=\frac{x^2}{1-x^2}$

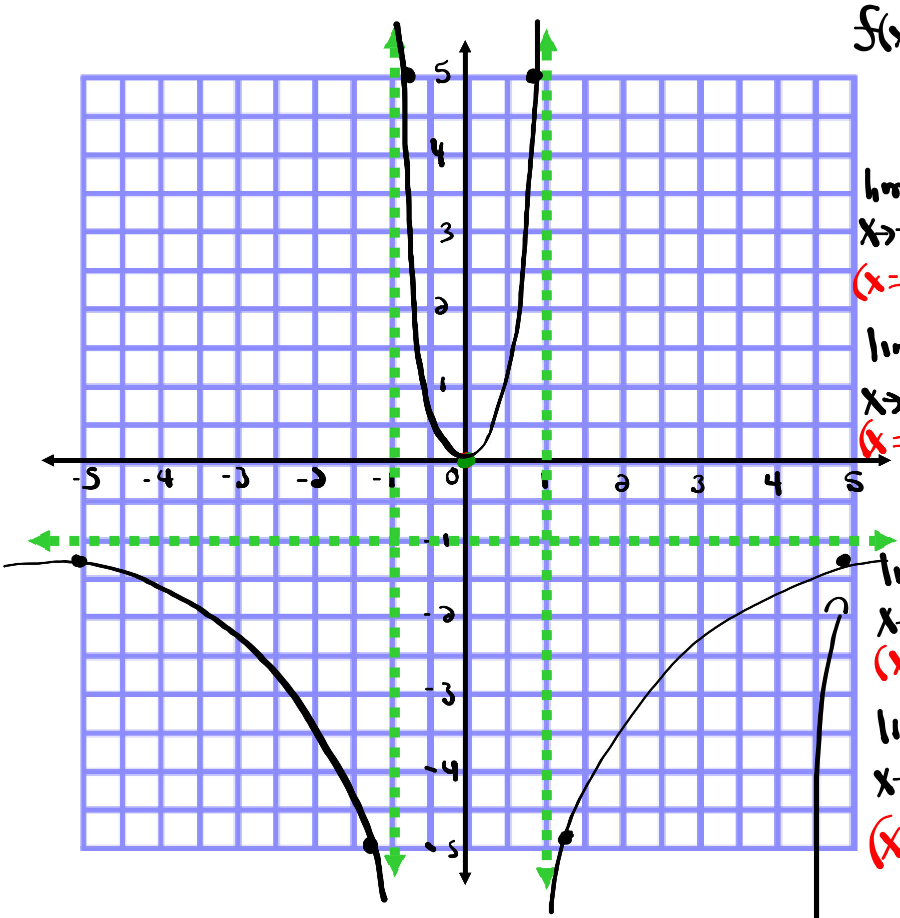
$f(-1)=\frac{(-1)^2}{1-(-1)^2}=\frac{1}{0}=$ und.

when x=1

$f(x)=\frac{x^2}{1-x^2}$

$f(1)=\frac{(1)^2}{1-(1)^2}=\frac{1}{0}=$ und

* No Inflection Points $x=\pm1$ are the Vertical Asymptotes.



$f(x)=\dfrac{x^2}{(1-x)(1+x)}$

$\lim\limits_{x\to -1^-} f(x) = \dfrac{(+)}{(+)(-)} = -\infty$

(x=-1.1)

$\lim\limits_{x\to -1^+} f(x) = \dfrac{(+)}{(+)(+)} = +\infty$

(x=-0.9)

$\lim\limits_{x\to 1^-} f(x) = \dfrac{(+)}{(+)(+)} = +\infty$

(x=0.9)

$\lim\limits_{x\to 1^+} f(x) = \dfrac{(+)}{(-)(+)} = -\infty$

(x=1.1)

$f(5) = \dfrac{(5)^2}{1-(5)^2} = \dfrac{25}{-24}$

$$f(x) = \frac{x^2}{1-x^2}$$

$$f'(x) = \frac{2x(1-x^2) + 2x(x^2)}{(1-x^2)^2}$$

$$f'(x) = \frac{2x - 2x^3 + 2x^3}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

$$f''(x) = \frac{2(1-x^2)^2 - 2x(2)(1-x^2)(-2x)}{(1-x^2)^4}$$

$$f''(x) = \frac{2(1-x^2)^2 + 8x^2(1-x^2)}{(1-x^2)^4}$$

$$f''(x) = \frac{2(1-x^2)\left[(1-x^2)+4x^2\right]}{(1-x^2)^4} =$$

$$f''(x) = \frac{2\cancel{(1-x^2)}(1+3x^2)}{(1-x^2)^{\cancel{4}\,3}} = \frac{2(1+3x^2)}{(1-x^2)^3}$$

Review:

① $f(x)=\frac{8(x-2)}{x^2}$ $f'(x)=\frac{-8(x-4)}{x^3}$ $f''(x)=\frac{16(x-6)}{x^4}$

① Intercepts:

x-int (y=0)

$\frac{0}{1}=\frac{8(x-2)}{x^2}$

$8(x-2)=0$

$x-2=0$

$x=2$

(2,0)

y-int (x=0)

$f(0)=\frac{8(0-2)}{(0)^2}$

$f(0)=\frac{-16}{0}$

$f(0)=$ undefined

No y-int

② Symmetry:

$f(x)=\frac{8(x-2)}{x^2}=\frac{8x-16}{x^2}$

$f(-x)=\frac{8(-x-2)}{(-x)^2}$

$f(-x)=\frac{8(-x-2)}{x^2}$

$f(-x)=\frac{-8(x+2)}{x^2}=\frac{-8x-16}{x^2}$

No symmetry

③ VA: (zeros of denom)

$f(x)=\frac{8(x-2)}{x^2}$

$x^2=0$

$\boxed{x=0}$

Check behaviour of f(x) near VA:

$\lim_{x\to 0^-} f(x)=\frac{(-)}{(+)}=-\infty$

(x = -0.01)

$\lim_{x\to 0^+} f(x)=\frac{(-)}{(+)}=-\infty$

(x = 0.01)

④ HA:

$f(x)=\frac{8(x-2)}{x^2}=\frac{8x-16}{x^2}$

$\lim_{x\to\infty}\frac{8x-16}{x^2}=0$

$\boxed{y=0}$

⑤ Intervals of Inc/Dec

$f'(x)=\frac{-8(x-4)}{x^3}$

CV:

$-8(x-4)=0$

$x-4=0$

$x=4$

$x^3=0$

$x=0$

min max

− + −

(-1) 0 (1) 4 (5)

Increasing on (0,4)

Decreasing on $(-\infty,0)$ + $(4,\infty)$

⑥ Max/Min:

$f(x)=\frac{8(x-2)}{x^2}$

$f(0)=\frac{8(0-2)}{(0)^2}=$ undefine

no min

x=0 is the VA

$f(4)=\frac{8(4-2)}{(4)^2}=\frac{16}{16}=1$

max at (4,1)

⑦ Concavity:

$f''(x)=\frac{16(x-6)}{x^4}$

CV:

$16(x-6)=0$

$x-6=0$

$x=6$

$x^4=0$

$x=0$

I.P.

− − +

(-1) 0 (1) 6 (7)

CU on $(6,\infty)$

CD on $(-\infty,0)$ + $(0,6)$

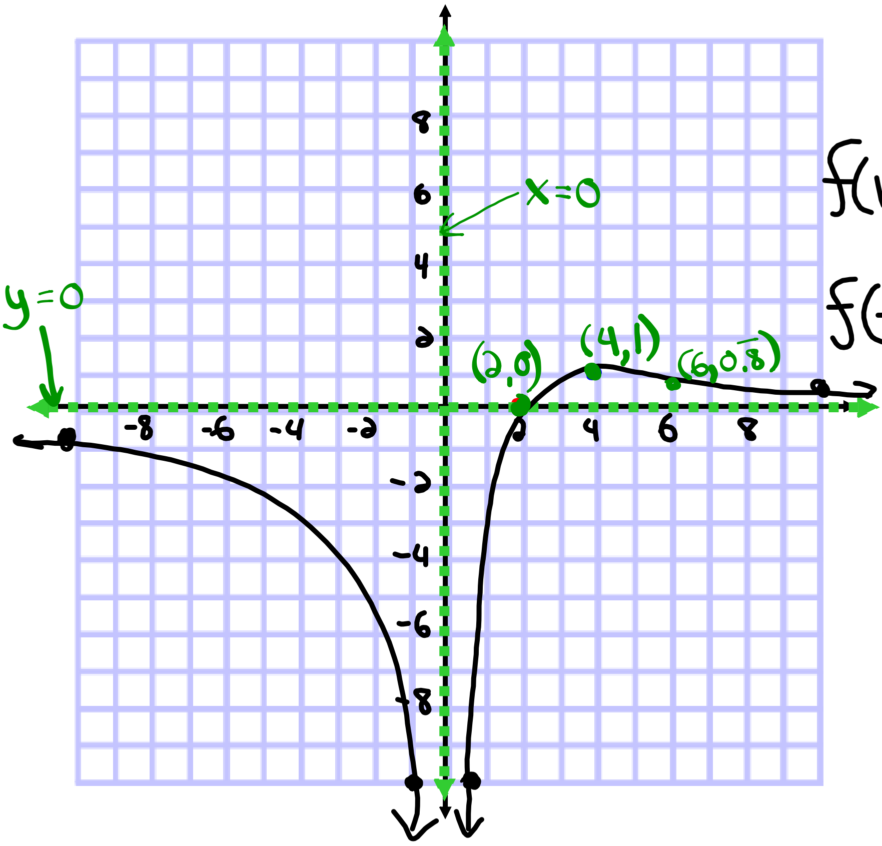
⑧ Inflection Point

$f(x)=\frac{8(x-2)}{x^2}$

$f(6)=\frac{8(6-2)}{6^2}=\frac{32}{36}$

$(6,0.\overline{8})$

Review:



$f(10)=0.64$

$f(-10)=-0.64$

Review:

Examine the function $f(x)=x^4-4x^3$ with respect to...

- Intercepts
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

$f(x)=x^4-4x^3$ $\quad f'(x)=4x^3-12x^2$ $\quad f''(x)=12x^2-24x$

$f(x)=x^3(x-4)$ $\quad f'(x)=4x^2(x-3)$ $\quad f''(x)=12x(x-2)$

① x-int $(y=0)$

$f(x)=x^3(x-4)$

$0=x^3(x-4)$

$x^3=0$ | $x-4=0$

$x=0$ | $x=4$

$(0,0)$ | $(4,0)$

② y-int $(x=0)$

$f(x)=x^4-4x^3$

$f(0)=(0)^4-4(0)^3$

$f(0)=0$

$(0,0)$

③ Intervals of Inc/Dec.

$f'(x)=4x^2(x-3)$

$4x^2=0$ | $x-3=0$

$x^2=0$ | $x=3$

$x=0$

CV: $x=0,3$

min

$-$ $\quad -$ $\quad +$

(-1) $\quad 0$ $\quad (1)$ $\quad 3$ $\quad (4)$

Decreasing on $(-\infty,3)$

Increasing on $(3,\infty)$

④ min @ $x=3$

$f(x)=x^4-4x^3$

$f(3)=(3)^4-4(3)^3$

$f(3)=81-108$

$f(3)=-27$

$(3,-27)$

⑤ No local max

⑥ Intervals of Concavity:

$f''(x)=12x(x-2)$

$12x=0$ | $x-2=0$

$x=0$ | $x=2$

CV: $x=0,2$

I.P. $\quad$ I.P.

$+$ $\quad -$ $\quad +$

(-1) $\quad 0$ $\quad (1)$ $\quad 2$ $\quad (3)$

CD on $(0,2)$

CU on $(-\infty,0)$ and $(2,\infty)$

⑦ Inflection points:

when $x=0$

$f(x)=x^4-4x^3$

$f(0)=(0)^4-4(0)^3$

$f(0)=0$

$(0,0)$

when $x=2$

$f(x)=x^4-4x^3$

$f(2)=(2)^4-4(2)^3$

$f(2)=16-32$

$f(2)=-16$

$(2,-16)$

Review:

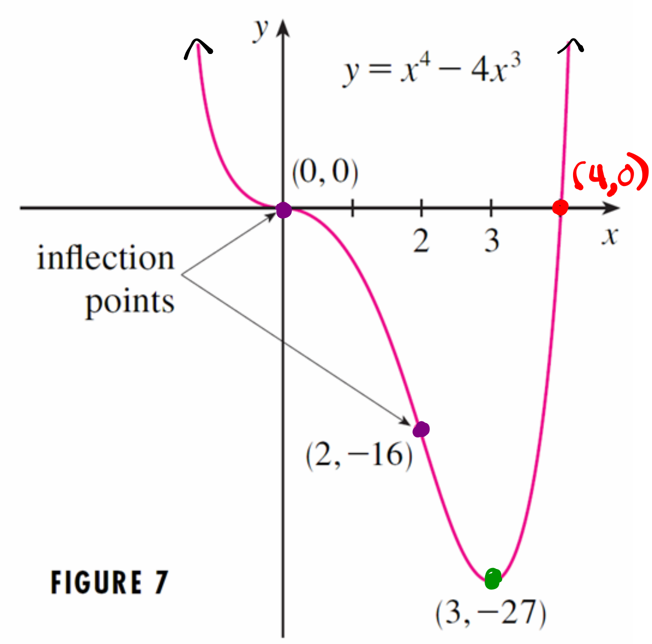


FIGURE 7

homework

Examine the function $f(x)=\frac{x^2}{x-7}$ with respect to...

- Intercepts
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

① x-int (y=0)

$f(x)=\frac{x^2}{x-7}$

$(x-7)\,0=\frac{x^2}{\cancel{x-7}}\,\cancel{(x-7)}$

$0=x^2$

$0=x$

$(0,0)$

② y-int (x=0)

$f(x)=\frac{x^2}{x-7}$

$f(0)=\frac{(0)^2}{(0)-7}=\frac{0}{-7}=0$

$y=0$

$(0,0)$

③ Symmetry:

$f(x)=\frac{x^2}{x-7}$

$f(-x)=\frac{(-x)^2}{(-x)-7}$

$f(-x)=\frac{x^2}{-x-7}$

No symmetry

④ VA: (denom = 0)

$x-7=0$

$x=7$

$\lim_{x\to7^-}\frac{x^2}{x-7}=\frac{(+)}{(-)}=-\infty$ (6.9)

$\lim_{x\to7^+}\frac{x^2}{x-7}=\frac{(+)}{(+)}=+\infty$ (x=7.1)

⑤ SA:

$$\begin{array}{r} x+7 \\ x-7\,\overline{)\,x^2} \\ -(x^2-7x) \\ 7x \\ -(7x-49) \\ 49\,R \end{array}$$

$y=x+7$

$m=\frac{1}{1}\ \frac{\text{rise}}{\text{run}}$

$b=7$ y-int

⑥ Intervals of Inc/Dec:

$F'(x)=\frac{x(x-14)}{(x-7)^2}$

max, neither, min

$+\ \ -\ \ -\ \ +$

(-1) 0 (1) 7 (8) 14 (15)

CV: $x=0$ | $x-14=0$, $x=14$ | $(x-7)^2=0$, $x-7=0$, $x=7$

Increasing on $(-\infty,0)+(14,\infty)$

$x<0$ + $x>14$

Decreasing on $(0,14)$

$0<x<14$

⑦ Local max/min

$f(x)=\frac{x^2}{x-7}$

when $x=0$

$f(0)=\frac{(0)^2}{(0)-7}=\frac{0}{-7}=0$

$(0,0)$

local max @ $(0,0)$

When $x=14$

$f(14)=\frac{(14)^2}{(14)-7}=\frac{196}{7}=28$

$(14,28)$

local min @ $(14,28)$

⑧ Intervals of Concavity:

$F''(x)=\frac{98}{(x-7)^3}$

I.P.

$-\ \ +$

(6) (7) (8)

CV: $98\neq0$ | $(x-7)^3=0$, $x-7=0$, $x=7$

Concave down on $(-\infty,7)$

$x<7$

Concave up on $(7,\infty)$

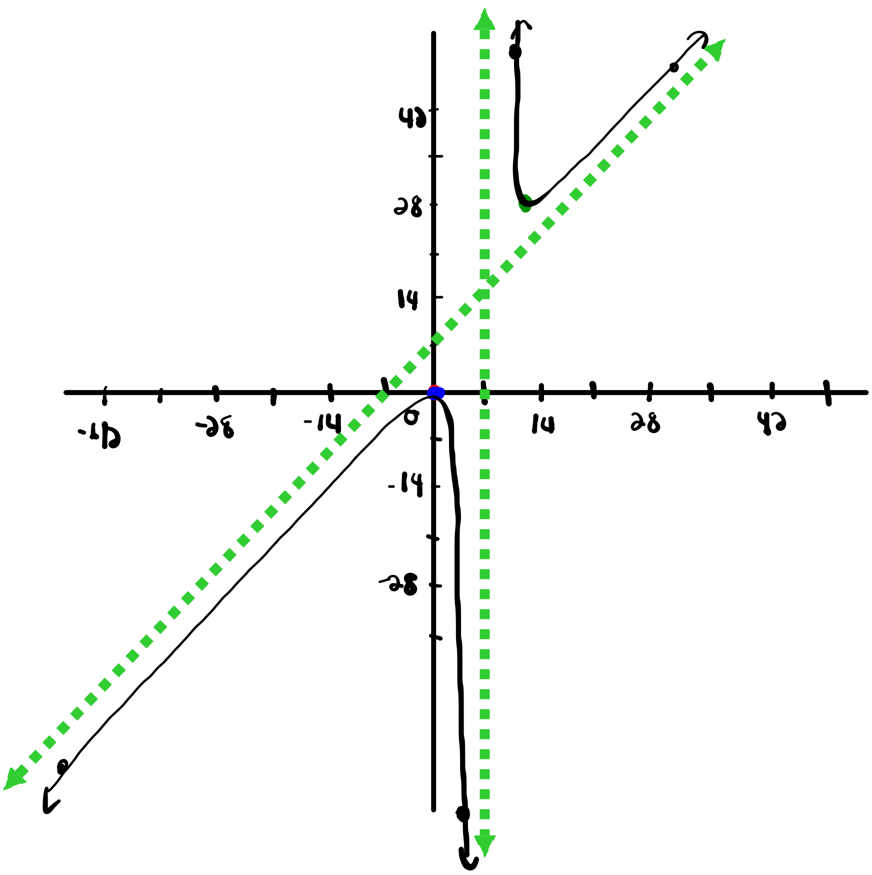
$x>7$

⑨ I.P. $(x=7)$

$f(x)=\frac{x^2}{x-7}$

$f(7)=\frac{(7)^2}{(7)-7}=\frac{49}{0}=\text{DNE}$

$x=7$ is the vertical asymptote



42
28
14
-42
-28
-14
0
14
28
42
-14
-28

Review:

$f(x) = x^4 - 3x^3 + 3x^2 - x$

$(1)^3 - 3(1)^2 + 3(1) - 1$
$1 - 3 + 3 - 1$

① x-int $(y=0)$

$0 = x^4 - 3x^3 + 3x^2 - x$

$0 = x(x^3 - 3x^2 + 3x - 1)$ syn. sub.

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 3 & -1 \\ & & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & \end{array}$$

$0 = x(x-1)(x^2 - 2x + 1)$ simple tri.

$-1 \times -1 = 1$
$-1 + -1 = -2$

$0 = x(x-1)(x-1)(x-1)$

$x = 0$ | $x - 1 = 0$, $x = 1$

$(0,0)$ | $(1,0)$

② y-int $(x=0)$

$f(x) = x^4 - 3x^3 + 3x^2 - x$

$f(0) = (0)^4 - 3(0)^3 + 3(0)^2 - (0) = 0$

$(0,0)$

Intervals of Inc/Dec:

$f(x) = x^4 - 3x^3 + 3x^2 - x$

$f'(x) = 4x^3 - 9x^2 + 6x - 1$ syn. sub

$4(1)^3 - 9(1)^2 + 6(1) - 1$
$4 - 9 + 6 - 1$

$$\begin{array}{r|rrrr} 1 & 4 & -9 & 6 & -1 \\ & & 4 & -5 & 1 \\ \hline & 4 & -5 & 1 & \end{array}$$

$f'(x) = (x-1)(4x^2 - 5x + 1)$ Hard - Trinomial

$f'(x) = (x-1)(x - \frac{1}{4})(x - \frac{4}{4})$

$-1 \times -4 = 4$
$-1 + -4 = -5$

$f'(x) = (x-1)(4x-1)(x-1)$

$f'(x) = (x-1)^2(4x-1)$

min, neither

$-$ $+$ $+$

(0) $\frac{1}{4}$ (0.5) 1 (2)

CV: $(x-1)^2 = 0$ | $4x - 1 = 0$
$x - 1 = 0$ | $4x = 1$
$x = 1$ | $x = \frac{1}{4}$

$x = \frac{1}{4}, 1$

Decreasing on $(-\infty, \frac{1}{4})$
$x < \frac{1}{4}$

Increasing on $(\frac{1}{4}, \infty)$
$x > \frac{1}{4}$