# Questions from Homework

① b) $\sum_{i=1}^{4} \frac{i}{4} f(i)$

$= \frac{1}{4} f(1) + \frac{2}{4} f(2) + \frac{3}{4} f(3) + \frac{4}{4} f(4)$

$= \frac{1}{4} f(1) + \frac{1}{2} f(2) + \frac{3}{4} f(3) + f(4)$

c) $\sum_{i=1}^{n} \frac{3}{n} f\left(1 + \frac{3}{4} i\right)$

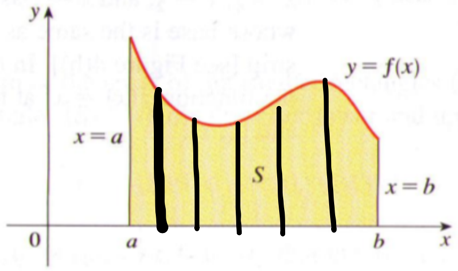
$= \frac{3}{n} f\left(1 + \frac{3}{4}(1)\right) + \frac{3}{n} f\left(1 + \frac{3}{4}(2)\right) + \frac{3}{n} f\left(1 + \frac{3}{4}(3)\right) + \ldots + \frac{3}{n} f\left(1 + \frac{3}{4}(n)\right)$

**Differential Calculus** arose from the tangent problem (wanting to investigate the slope at each point along the curve).... whereas **Integral Calculus** arose from a seemingly unrelated problem, the area problem! (wanting to find the area under a curve)

Today, we are going to look at approximating the area under a curve.

### The Area Problem

We begin by attempting to solve the *area problem:* Find the area of the region $S$ that lies under the curve $y = f(x)$ from $a$ to $b$. This means that $S$, illustrated in Figure 1, is bounded by the graph of a continuous function $f$ [where $f(x) \geq 0$], the vertical lines $x = a$ and $x = b$, and the $x$-axis.




However, it is not so easy to find the area of a region with curved sides exactly so our goal is to find a way to approximate the area using our knowledge of limits. And perhaps we will arrive at an exact definition.

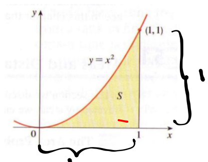
So, thinking back to when we developed the definition of a derivative...we approximated the slope of the tangent line by moving one of the points on the secant line closer and closer to the other so that $\Delta h$ approached 0.

Similarly, we use the same idea for approximating areas. We will approximate the region S under the curve by rectangles and then we will take the limit of the areas of these rectangles as we increase the number of rectangles.

example:

Let's approximate the area under the curve for $y = x^2$ betweeen $x = 0$ and $x = 1$.

$a = 0$ $b = 1$




We may first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, we can also notice that the area of S is most definitely less than 0.5, but we can do much better than that!

Suppose we divide S into four equal strips, $S_1$, $S_2$, $S_3$, and $S_4$ by drawing vertical lines x=1/4, x=1/2, and x=3/4 we have:

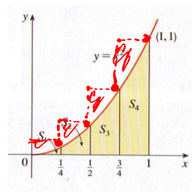



FIGURE 4 (a)

We can then approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip. (in other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the right endpoints of the subintervals.)

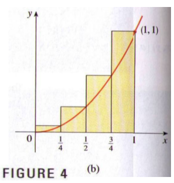



FIGURE 4 (b)

Each rectangle has width $\frac{1}{4}$ and the heights are $(\frac{1}{4})^2$, $(\frac{1}{2})^2$, $(\frac{3}{4})^2$, and $1^2$. If we let $R_4$ be the sum of the areas of these approximating rectangles, we get

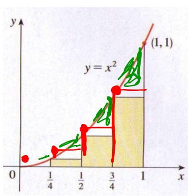
$$R_4 = \tfrac{1}{4}\cdot(\tfrac{1}{4})^2 + \tfrac{1}{4}\cdot(\tfrac{1}{2})^2 + \tfrac{1}{4}\cdot(\tfrac{3}{4})^2 + \tfrac{1}{4}\cdot 1^2 = \tfrac{15}{32} = 0.46875$$

From Figure 4(b) we see that the area $A$ of $S$ is less than $R_4$, so **(overshot)**

$$A < 0.46875$$

We can also approximate each strip by using smaller rectangles whose base is the same as the strip and whose height is the same as the left edge of the strip. (in other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the left-hand endpoints of the subintervals.)

The sum of the areas of these approximating rectangles is:




(The leftmost rectangle has collapsed because its height is 0.)

$L_4 = 7/32 = 0.21875$ scan in sum*******

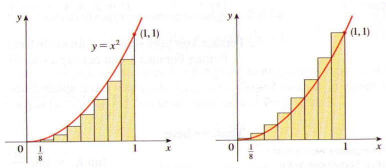
**(undershot)**

We see that the area of $S$ is larger than $L_4$, so we have lower and upper estimates for $A$:

$$\boxed{0.21875 < A < 0.46875}$$

We can repeat this procedure with a larger number of strips. Figures below shows what happen when we divide the region S into eight strips of equal width....

(we are narrowing in on the true area)



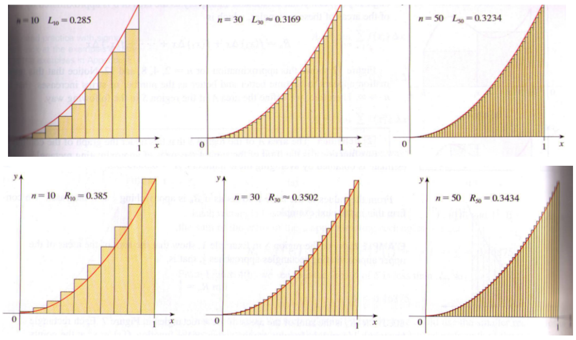

(a) Using left endpoints (b) Using right endpoints

By computing the sum of the areas of the smaller rectangles ($L_8$) and the sum of the areas of the larger rectangles ($R_8$), we obtain better lower and upper estimates for $A$:

$$0.2734375 < A < 0.3984375$$

So one possible answer to the question is to say that the true area of $S$ lies somewhere between 0.2734375 and 0.3984375.

What are the sums of the areas of the rectangles as the number of rectangles gets large? (increasing the number of strips)
Here are the sums for the areas of the rectangles using both left hand ($L_n$) and right hand ($R_n$) endpoints for n = 10, n = 30 and n = 50 strips....



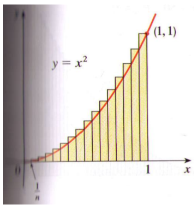

sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \boxed{\frac{1}{3}}$$

Using a computer software to compute the sum of the areas of the rectangle using both left hand and right hand endpoints we have.....as the number of strips, n, gets larger......

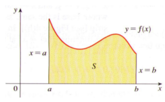
| $n$ | $L_n$ | $R_n$ |
|---|---|---|
| 10 | 0.2850000 | 0.3850000 |
| 20 | 0.3087500 | 0.3587500 |
| 30 | 0.3168519 | 0.3501852 |
| 50 | 0.3234000 | 0.3434000 |
| 100 | 0.3283500 | 0.3383500 |
| 1000 | 0.3328335 | 0.3338335 |

$\lim_{n\to\infty} L_n = \frac{1}{3}$ $\lim_{n\to\infty} R_n = \frac{1}{3}$

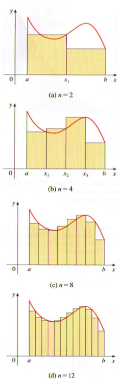


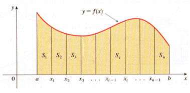

We could obtain better estimates by increasing the number of strips. The table at the left shows the results of similar calculations (with a computer) using $n$ rectangles whose heights are found with left-hand endpoints ($L_n$) or right-hand endpoints ($R_n$). In particular, we see by using 50 strips that the area lies between 0.3234 and 0.3434. With 1000 strips we narrow it down even more: $A$ lies between 0.3328335 and 0.3338335. A good estimate is obtained by averaging these numbers: $A \approx 0.3333335$.

Looking back at the function shaped as.....




we can apply the same technique for approximating the area by summing the rectangles.....



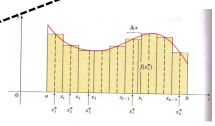


> **2 Definition** The **area** $A$ of the region $S$ that lies under the graph of the continuous function $f$ is the limit of the sum of the areas of approximating rectangles:
>
> $$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

$$\Delta x = \frac{b-a}{n}$$

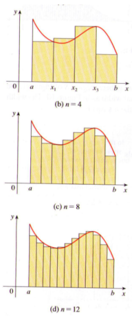
$$A = \lim_{n\to\infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

sample points $x_n$*



We often use sigma notation to write sums with many terms more compactly. For instance.........................

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$



# Riemann Sum

**2 Definition** The **area** $A$ of the region $S$ that lies under the graph of the continuous function $f$ is the limit of the sum of the areas of approximating rectangles:

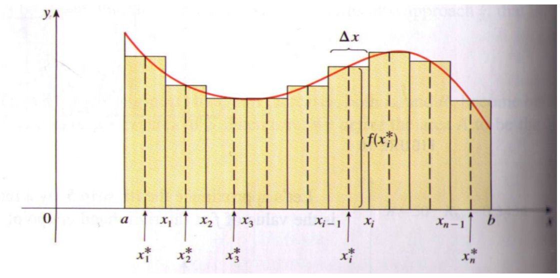
$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

It can be proved that the limit in Definition 2 always exists, since we are assuming $f$ is continuous. It can also be shown that we get the same value if we use left endpoints:

$$\boxed{3} \quad A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x]$$

In fact, instead of using left endpoints or right endpoints, we could take the height of the $i$th rectangle to be the value of $f$ at *any* number $x_i^*$ in the $i$th subinterval $[x_{i-1}, x_i]$. We call the numbers $x_1^*, x_2^*, \ldots, x_n^*$ the **sample points**. Figure 13 shows approximating rectangles when the sample points are not chosen to be endpoints. So a more general expression for the area of $S$ is

$$\boxed{4} \quad A = \lim_{n\to\infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$




We often use **sigma notation** to write sums with many terms more compactly. For instance,

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

The sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x \qquad \text{Riemann Sum}$$

that occurs in Definition 2 is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826–1866). We know that if $f$ happens to be positive, then the Riemann sum can be interpreted as a sum of areas of approximating rectangles (see Figure 1). By comparing Definition 2 with the definition of area in Section 5.1,

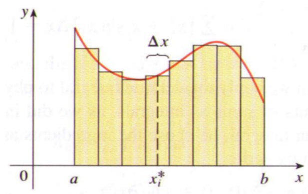



FIGURE 1
If $f(x) \geq 0$, the Riemann sum $\sum f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

So, how do we compute the area under a curve using a Riemann sum?? Let's go....

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

infinite rectangles $n \to \infty$

Recall the sigma formulas, we will need them here!
They're going to help... $\sum_{i=1}^{n} i^0 = 1^0 + 2^0 + 3^0 + 4^0 + \ldots n^0 = n$

$$\boxed{\sum_{i=1}^{n} i} = 1 + 2 + 3 + 4 + \ldots . n = \boxed{\frac{n(n+1)}{2}}$$

$$\boxed{\sum_{i=1}^{n} i^2} = 1^2 + 2^2 + 3^2 + 4^2 + \ldots . n^2 = \boxed{\frac{n(n+1)(2n+1)}{6}}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + \ldots n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

# Riemann Sum

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

base of rectangle

height

Evaluating the area under a curve y = f(x) from a to b.

$\Delta x$ (width of subinterval) will be the size of the interval [a,b] divided by n strips.

$$\Delta x = \frac{b-a}{n}$$

The $i^{th}$ position, $x_i$, is: $x_i^* = a + i\Delta x$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 = \left[\frac{n^2+n}{2}\right]\left[\frac{n^2+n}{2}\right]$$

$$= \frac{n^4+n^3+n^3+n^2}{4}$$

$$= \boxed{\frac{n^4+2n^3+n^2}{4}}$$

Example: Find the area under the curve for $f(x) = x^2 + 1$ between $x = 1$ and $x = 4$ using the limit as $n \to \infty$ of the Riemann sum.

$a=1$, $b=4$

$\boxed{\text{Area} = 24}$

① $\Delta x = \frac{b-a}{n}$

$\Delta x = \frac{4-1}{n}$

$\Delta x = \frac{3}{n}$

② $x_i^* = a + i\Delta x$

$x_i^* = 1 + i\left(\frac{3}{n}\right) = 1 + \frac{3i}{n}$

③ $f(x_i^*)$

$\left(1+\frac{3i}{n}\right)\left(1+\frac{3i}{n}\right)$

$1 + \frac{3i}{n} + \frac{3i}{n} + \frac{9i^2}{n^2}$

$f\left(1+\frac{3i}{n}\right) = \left(1+\frac{3i}{n}\right)^2 + 1$

$f\left(1+\frac{3i}{n}\right) = 1 + \frac{6i}{n} + \frac{9i^2}{n^2} + 1$

$f\left(1+\frac{3i}{n}\right) = 2 + \frac{6i}{n} + \frac{9i^2}{n^2}$

④ $A = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(2 + \frac{6i}{n} + \frac{9i^2}{n^2}\right)}_{f(x_i^*)} \underbrace{\left(\frac{3}{n}\right)}_{\Delta x}$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} + \frac{18i}{n^2} + \frac{27i^2}{n^3}$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} + \sum_{i=1}^{n} \frac{18i}{n^2} + \sum_{i=1}^{n} \frac{27i^2}{n^3}$

Quad Linear Constant

$A = \lim_{n\to\infty} \frac{27}{n^3}\boxed{\sum_{i=1}^{n} i^2} + \frac{18}{n^2}\boxed{\sum_{i=1}^{n} i} + \boxed{\sum_{i=1}^{n} \frac{6}{n}}$

$A = \lim_{n\to\infty} \frac{\cancel{27}^{\,9}}{\cancel{n^3}_{\,n^2}}\left[\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}_{\,2}}\right] + \frac{\cancel{18}^{\,9}}{\cancel{n^2}_{\,n}}\left[\frac{\cancel{n}(n+1)}{\cancel{2}_{\,1}}\right] + \frac{6}{\cancel{n}} \cdot \cancel{n}$

$A = \lim_{n\to\infty} \frac{9(n+1)(2n+1)}{2n^2} + \frac{9(n+1)}{n} + 6$

$9(2n^2+n+2n+1)$

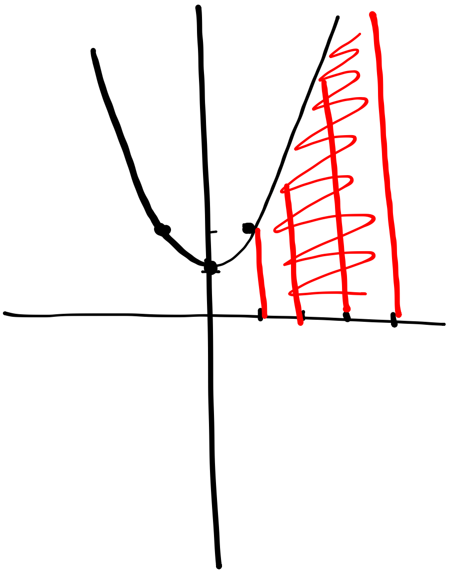
$A = \lim_{n\to\infty} \frac{9(n+1)(2n+1)}{2n^2} + \lim_{n\to\infty} \frac{9n+9}{n} + \lim_{n\to\infty} 6$

$A = \lim_{n\to\infty} \frac{18n^2+27n+9}{2n^2} + \lim_{n\to\infty} \frac{9n+9}{n} + \lim_{n\to\infty} 6$

$A = \frac{18}{2} + \frac{9}{1} + 6$

$A = 9 + 9 + 6$

$\boxed{A = 24}$



Find the area under the curve for $f(x) = x^2 - x$ between $x = 1$ and $x = 3$ using the limit as $n \to \infty$ of the Riemann sum.

$a = 1$, $b = 3$

$\boxed{\text{Area} = 14/3}$

① $\Delta x = \dfrac{b-a}{n}$

$\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$

② $x_i^* = a + i\Delta x$

$x_i^* = 1 + i\left(\dfrac{2}{n}\right)$

$x_i^* = 1 + \dfrac{2i}{n}$

③ $f(x_i^*)$

$f\left(1+\dfrac{2i}{n}\right) = \left(1+\dfrac{2i}{n}\right)^2 - \left(1+\dfrac{2i}{n}\right)$

$f\left(1+\dfrac{2i}{n}\right) = \cancel{1} + \cancel{\dfrac{2i}{n}} + \dfrac{2i}{n} + \dfrac{4i^2}{n^2} \cancel{-1} - \cancel{\dfrac{2i}{n}}$

$f\left(1+\dfrac{2i}{n}\right) = \dfrac{4i^2}{n^2} + \dfrac{2i}{n}$

④ $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

Step 3 Step 1

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\dfrac{4i^2}{n^2} + \dfrac{2i}{n}\right)\left(\dfrac{2}{n}\right)$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\dfrac{8i^2}{n^3} + \dfrac{4i}{n^2}\right)$

Quad linear

$A = \lim_{n\to\infty} \dfrac{8}{n^3}\boxed{\sum_{i=1}^{n} i^2} + \dfrac{4}{n^2}\boxed{\sum_{i=1}^{n} i}$

$A = \lim_{n\to\infty} \dfrac{\cancel{8}^{4}}{n^2\cancel{n^3}}\left[\dfrac{\cancel{n}(n+1)(2n+1)}{\cancel{6}_{3}}\right] + \dfrac{\cancel{4}^{2}}{n\cancel{n^2}}\left[\dfrac{\cancel{n}(n+1)}{\cancel{2}_{1}}\right]$

$A = \lim_{n\to\infty} \dfrac{4(n+1)(2n+1)}{3n^2} + \lim_{n\to\infty} \dfrac{2(n+1)}{n}$

$4(2n^2+3n+1)$

$A = \lim_{n\to\infty} \dfrac{8n^2+12n+4}{3n^2} + \lim_{n\to\infty} \dfrac{2n+2}{n}$

$A = \dfrac{8}{3} + 2$

$A = \dfrac{8}{3} + \dfrac{6}{3}$

$\boxed{A = \dfrac{14}{3}}$

Find the area under the curve for $f(x) = x + 1$ between $x = 1$ and $x = 3$ using the limit as $n \to \infty$ of the Riemann sum.

$a=1$ $\quad$ $b=3$

① $\Delta x = \frac{b-a}{n}$

$\Delta x = \frac{3-1}{n} = \frac{3}{n}$

② $x_i^* = a + i\Delta x$

$x_i^* = 1 + i\left(\frac{3}{n}\right)$

$x_i^* = 1 + \frac{3i}{n}$

③ $f(x_i^*)$

$f\left(1+\frac{3i}{n}\right) = \left(1+\frac{3i}{n}\right) + 1$

$f\left(1+\frac{3i}{n}\right) = \frac{3i}{n} + 2$

④ $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

Step 3 $\quad$ Step 1

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3i}{n} + 2\right)\left(\frac{3}{n}\right)$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{9i}{n^2} + \frac{6}{n}\right)$

Linear $\quad$ Constant

$A = \lim_{n\to\infty} \frac{9}{n^2}\boxed{\sum_{i=1}^{n} i} + \boxed{\sum_{i=1}^{n} \frac{6}{n}}$

$A = \lim_{n\to\infty} \frac{9}{\cancel{n^2}_{\,n}}\left[\frac{\cancel{n}^{\,1}(n+1)}{2}\right] + \frac{6}{\cancel{n}}\cdot\cancel{n}$

$A = \lim_{n\to\infty} \frac{9(n+1)}{2n} + \lim_{n\to\infty} 6$

$A = \lim_{n\to\infty} \frac{9n+9}{2n} + \lim_{n\to\infty} 6$

$A = \frac{9}{2} + 6$

$A = \frac{9}{2} + \frac{12}{2} = \boxed{\frac{21}{2}}$

Find the area under the curve for $f(x) = x^3 - 6x$ between $x = 0$ and $x = 3$ using the limit as $n \to \infty$ of the Riemann sum.

$a = 0$ $b = 3$

$\boxed{\text{Area} = -27/4}$

① $\Delta x = \frac{b-a}{n}$

$\Delta x = \frac{3-0}{n}$

$\Delta x = \frac{3}{n}$

② $x_i^* = a + i\Delta x$

$x_i^* = 0 + i\left(\frac{3}{n}\right) = \frac{3i}{n}$

③ $f(x_i^*)$

$f\left(\frac{3i}{n}\right) = \left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)$

$f\left(\frac{3i}{n}\right) = \frac{27i^3}{n^3} - \frac{18i}{n}$

④ $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{27i^3}{n^3} - \frac{18i}{n}\right)\left(\frac{3}{n}\right)$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{81i^3}{n^4} - \frac{54i}{n^2}$

Cubic Linear

$A = \lim_{n\to\infty} \frac{81}{n^4}\boxed{\sum_{i=1}^{n} i^3} - \frac{54}{n^2}\boxed{\sum_{i=1}^{n} i}$

$A = \lim_{n\to\infty} \frac{81}{n^4}\left[\left(\frac{n(n+1)}{2}\right)^2\right] - \frac{\cancel{54}^{27}}{n\,\cancel{n^2}}\left[\frac{\cancel{n}(n+1)}{\cancel{2}}\right]$

$A = \lim_{n\to\infty} \frac{81}{n^4}\left[\frac{n^4+2n^3+n^2}{4}\right] - \frac{27(n+1)}{n}$

$A = \lim_{n\to\infty} \frac{81n^4+162n^3+81n^2}{4n^4} - \lim_{n\to\infty} \frac{27n+27}{n}$

$A = \frac{81}{4} - 27$

$A = \frac{81}{4} - \frac{108}{4} = \boxed{\frac{-27}{4}}$

Find the area under the curve for $f(x) = 3x - x^2$ between $x = 0$ and $x = 5$ using the limit as $n \to \infty$ of the Riemann sum.

Area = - 25/6

## Practice

Red Book - Ex. 10.4 - pp. 474 #3