Warm Up

Find the area under the curve for $f(x) = x^3 - 6x$ between $x = 0$ and $x = 3$ using antiderivatives.

$a=0$ $b=3$ (definite integral)

In practice we choose the antiderivative where C = 0

Area = - 27/4

$$A = \int_0^3 x^3 - 6x \, dx = \frac{x^4}{4} - \frac{6x^2}{2} \Big]_0^3$$

$$= \frac{x^4}{4} - 3x^2 \Big]_0^3$$

$$= \frac{(3)^4}{4} - 3(3)^2 - \left[\frac{(0)^4}{4} - 3(0)^2\right]$$

$$= \frac{81}{4} - 27 - (0-0)$$

$$= \frac{81}{4} - \frac{108}{4}$$

$$= -\frac{27}{4}$$

The Definite Integral

When we computed the area under a curve by summing the areas of many rectangles, the limit took the form....

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

(this also arises when we try to find the distance traveled by an object)

It turns out that this same type of limit occurs in a wide variety of situations even when f is not necessarily a positive function. Limits of the this form also arise in finding lengths of curves, volumes of solids, centers of mass, force due to water pressure, and work, as well as other quantities. We therefore give this type of limit a special name and notation.

2 Definition of a Definite Integral If f is a continuous function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. We let $x_0 \, (=a), x_1, x_2, \dots, x_n \, (=b)$ be the endpoints of these subintervals and we choose **sample points** $x_1^*, x_2^*, \dots, x_n^*$ in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

Definite Integral $\boxed{\int_a^b f(x)\,dx} = \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x}$ Reimann Sum

NOTE 1 ◻ The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and was chosen because an integral is a limit of sums. In the notation $\int_a^b f(x)\,dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**. The symbol dx has no official meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol. The procedure of calculating an integral is called **integration**.

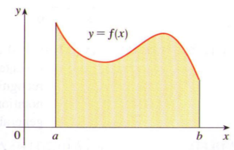

FIGURE 2
If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

Indefinite Integral

$$\int x^3\,dx = \boxed{\frac{x^4}{4} + C}$$

Definite Integral

$$\int_{-1}^{2} x^3\,dx = \frac{x^4}{4}\Big]_{-1}^{2}$$

$\leftarrow b$

$\swarrow a$

$$= \frac{(2)^4}{4} - \frac{(-1)^4}{4}$$

$$= 4 - \frac{1}{4}$$

$$= \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$$

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

* In practice we choose the antiderivative where C = 0 *

Use the FTC to evaluate each of the following definite integrals:

$$\int_{-1}^{2} x^3\,dx = \frac{x^4}{4}\Big]_{-1}^{2}$$

b a

$$= \frac{(2)^4}{4} - \frac{(-1)^4}{4}$$

$$= 4 - \frac{1}{4}$$

$$= \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$$

$$\int_2^5 (2x^3 - 3x^2 + 7x + 2)\,dx$$

$$= \frac{2x^4}{4} - \frac{3x^3}{3} + \frac{7x^2}{2} + 2x \Big]_2^5$$

$$= \frac{x^4}{2} - x^3 + \frac{7x^2}{2} + 2x \Big]_2^5$$

$$= \frac{(5)^4}{2} - (5)^3 + \frac{7(5)^2}{2} + 2(5) - \left[\frac{(2)^4}{2} - (2)^3 + \frac{7(2)^2}{2} + 2(2)\right]$$

$$= \underbrace{\frac{625}{2} - 125 + \frac{175}{2} + 10}_{} - [\underbrace{8 - 8 + 14 + 4}_{}]$$

$$= 285 - 18$$

$$= 267$$

$$\int_1^8 \frac{1}{\sqrt[3]{x^2}}\,dx$$

$$= \int_1^8 \frac{1}{x^{2/3}}\,dx \qquad \frac{x^{1/3}}{1/3} = x^{1/3} \cdot \frac{3}{1} = 3x^{1/3}$$

$$= \int_1^8 x^{-2/3}\,dx = 3x^{1/3}\Big]_1^8$$

$$= 3(8)^{1/3} - 3(1)^{1/3}$$

$$= 3(2) - 3(1)$$

$$= 6 - 3$$

$$= 3$$

$$\int_1^4 \frac{t^2+\sqrt{t}-2}{t}\,dt$$

$$\int_1^4 t + t^{-1/2} - \frac{2}{t}\,dt = \frac{t^2}{2} + 2t^{1/2} - 2\ln t \Big]_1^4$$

$$= \frac{(4)^2}{2} + 2(4)^{1/2} - 2\ln 4 - \left[\frac{(1)^2}{2} + 2(1)^{1/2} - 2\boxed{\ln 1}\right]$$

$\ln 1 = 0$

$$= 8 + 4 - 2\ln 4 - \frac{1}{2} - 2 + 0$$

$$= \frac{19}{2} - 2\ln 4$$

$$\int_\pi^{2\pi} \sin x\,dx = -\cos x \Big]_\pi^{2\pi}$$

$$= -\cos(2\pi) - [-\cos(\pi)]$$

$$= -1 + -1$$

$$= -2$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1} x \Big]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}$$

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus: differential calculus and integral calculus. Differential calculus arose from the tangent problem, whereas integral calculus arose from the seemingly unrelated problem, the area problem. Newton's teacher at Cambridge, Isaac Barrow (1630-1677), discovered that these two problems are actually closely related.

In fact, he realized that differentiation and integration are inverse processes. The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral. It was Newton and Leibniz who exploited this relationship and used it to develop calculus into a systematic mathematical method. In particular, they saw that the Fundamental Theorem enabled them to compute areas and integrals very easily without having to compute them as limits as we have been doing in the past couple of days.

(Stewart 4th Ed. - pp. 391)

Use the Fundamental Theorem of Calculus to find the area under the graph of:

$f(x) = x^2$ between $x = 0$ and $x = 1$

$$A = \int_0^1 x^2 dx \quad = \frac{x^3}{3}\Big|_0^1$$

$$A = \frac{1}{3}$$

$$= \frac{(1)^3}{3} - \frac{(0)^3}{3}$$

$$= \frac{1}{3}$$

Use the Fundamental Theorem of Calculus to find the area under the graph of:

$f(x) = 2x^2 - 3x + 2$ between $x = 0$ and $x = 2$

$$A = \int_0^2 (2x^2 - 3x + 2)dx$$

$$A = \frac{10}{3}$$

$$= \frac{2x^3}{3} - \frac{3x^2}{2} + 2x \Big|_0^2$$

$$= \frac{2(2)^3}{3} - \frac{3(2)^2}{2} + 2(2) - \left[\frac{2(0)^3}{3} - \frac{3(0)^2}{2} + 2(0)\right]$$

$$= \frac{16}{3} - 6 + 4 - 0$$

$$= \frac{16}{3} - 2$$

$$= \frac{16}{3} - \frac{6}{3}$$

$$= \frac{10}{3}$$

Use the Fundamental Theorem of Calculus to find the area under the curves:

$$A=\int_2^4 (x^2+2)dx = \frac{x^3}{3}+2x\Big]_2^4$$

$$= \frac{(4)^3}{3}+2(4)-\left[\frac{(2)^3}{3}+2(2)\right]$$

$$= \frac{64}{3}+8-\frac{8}{3}-4$$

$$= \frac{56}{3}+\frac{24}{3}-\frac{12}{3} = \frac{68}{3}$$

$$A=\int_0^{\frac{\pi}{2}} (\sin x)dx = -\cos x\Big]_0^{\pi/2}$$

$$= -\cos\left(\frac{\pi}{2}\right)-\left[-\cos(0)\right]$$

$$= 0+1$$

$$= 1$$

$$A=\int_1^3 e^x dx = e^x\Big]_1^3$$

$$= e^3 - e^1$$

$$= e(e^2-1)$$

Be careful, what about this one??? Why is it different?? Does the FTC apply???

$$\cancel{A=-\frac{4}{3}}$$

$$A=\int_{-1}^3 \frac{1}{x^2}dx = \int_{-1}^3 x^{-2}dx = \frac{x^{-1}}{-1}\Big]_{-1}^3 = -\frac{1}{x}\Big]_{-1}^3$$

$$= \frac{-1}{3}-\left(\frac{-1}{-1}\right)$$

$$= -\frac{1}{3}-1$$

$$= -\frac{1}{3}-\frac{3}{3} = -\frac{4}{3}$$

The FTC applies only to continuous functions over the interval [a,b]. Since there is infinite discontinuity at x = 0, it is not continuous on [-1,3]. This integral does not exist. We cannot compute the area!!!

Homework

Exercise 10.1 - Red Book - pp. 454 #1

Exercise 11.2 - Red Book - pp. 505 #1 - 4

$$\int_1^2 (x^2 - 3)\,dx \qquad A = -\frac{2}{3}$$

$$\int_0^1 (4t+1)^2\,dt \qquad A = \frac{31}{3}$$

$$\int_1^4 3\sqrt{x}\,dx \qquad A = 14$$

$$\int_{-8}^{-1} \frac{x + 2x^2}{\sqrt[3]{x}}\,dx \qquad A = -\frac{3453}{20} = -172.65$$

$$A = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin x)\,dx \qquad A = \frac{\sqrt{3}}{2}$$

$$A = \int_{-2}^{2} (4 - x^2)\,dx \qquad A = \frac{32}{3}$$

$$A = \int_0^{\frac{\pi}{2}} (\cos x)\,dx \qquad A = 1$$

$$A = \int_3^6 \frac{dx}{x} \qquad A = \ln 2$$