

# Making a Complete Sketch

**Example:**

Examine the function  $f(x) = 3x^5 - 5x^3$  with respect to...

- Intercepts  $f(x)$
- ~~Symmetry~~
- Asymptotes (No asymptotes for polynomial functions)
- Intervals of Increase or Decrease  $f'(x)$
- Local Maximum and Minimum values  $f(x)$
- $f''(x)$  Concavity and Points of Inflection  $f(x)$
- Sketch the Curve

$$f(x) = 3x^5 - 5x^3 \quad f'(x) = 15x^4 - 15x^2 \quad f''(x) = 60x^3 - 30x$$

$$f(x) = x^3(3x^2 - 5) \quad f'(x) = 15x^2(x^2 - 1) \quad f''(x) = 30x(x^2 - 1)$$

$$f'(x) = 15x^2(x-1)(x+1)$$

① x-int ( $y=0$ )

$$f(x) = x^3(3x^2 - 5)$$

$$0 = x^3(3x^2 - 5)$$

$$x^3 = 0 \quad 3x^2 - 5 = 0$$

$$x = 0 \quad 3x^2 = 5$$

$$(0,0) \quad x^2 = \frac{5}{3}$$

$$x = \pm\sqrt{\frac{5}{3}}$$

$$(1.29, 0) \quad + (-1.29, 0)$$

② y-int ( $x=0$ )

$$f(x) = 3x^5 - 5x^3$$

$$f(0) = 3(0)^5 - 5(0)^3$$

$$f(0) = 0$$

$$(0,0)$$

③ Intervals of Inc/Dec.

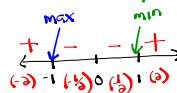
$$f'(x) = 15x^2(x-1)(x+1)$$

$$0 = 15x^2(x-1)(x+1)$$

$$15x^2 = 0 \quad x-1 = 0 \quad x+1 = 0$$

$$x^2 = 0 \quad x = 1 \quad x = -1$$

$$x = 0 \quad x = 1 \quad x = -1$$



Increasing on  $(-\infty, -1) + (1, \infty)$   
 Decreasing on  $(-1, 0) + (0, 1)$   
 or  $(-1, 1)$

CV:  $x = -1, 0, 1$

④ max @  $x = -1$

$$f(x) = 3x^5 - 5x^3$$

$$f(-1) = 3(-1)^5 - 5(-1)^3$$

$$f(-1) = -3 + 5$$

$$f(-1) = 2$$

$$(-1, 2)$$

⑤ min @  $x = 1$

$$f(x) = 3x^5 - 5x^3$$

$$f(1) = 3(1)^5 - 5(1)^3$$

$$f(1) = 3 - 5$$

$$f(1) = -2$$

$$(1, -2)$$

⑥ Intervals of Concavity:

$$f''(x) = 30x(x^2 - 1)$$

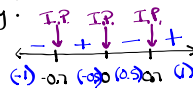
$$0 = 30x(x^2 - 1)$$

$$30x = 0 \quad x^2 - 1 = 0$$

$$x = 0 \quad x^2 = 1$$

$$x = \pm\sqrt{1}$$

$$x = \pm 1$$



CD on  $(-\infty, -0.7) + (0, 0.7)$   
 CU on  $(-0.7, 0) + (0.7, \infty)$

CV:  $x = -0.7, 0, 0.7$

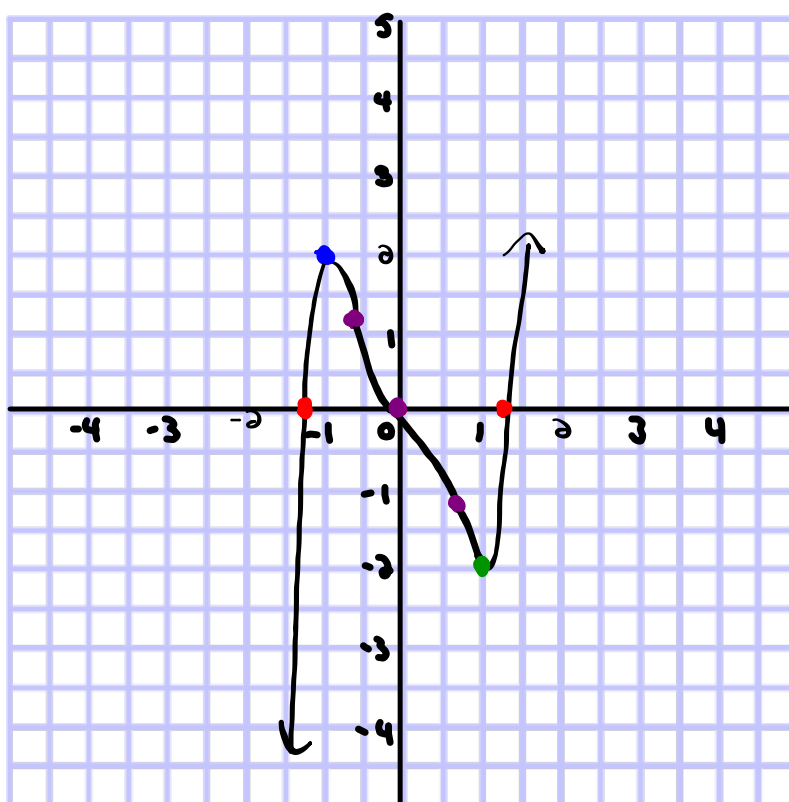
⑦ Inflection Points

$$f(x) = 3x^5 - 5x^3$$

$$f(-0.7) = 3(-0.7)^5 - 5(-0.7)^3 = -0.504 + 1.715 = 1.2 \quad (-0.7, 1.2)$$

$$f(0) = 3(0)^5 - 5(0)^3 = 0 - 0 = 0 \quad (0, 0)$$

$$f(0.7) = 3(0.7)^5 - 5(0.7)^3 = 0.504 - 1.715 = -1.2 \quad (0.7, -1.2)$$



Assignment:

$$f(x) = x^2 + x^3 \quad f'(x) = 2x + 3x^2 \quad f''(x) = 2 + 6x$$

$$f(x) = x^2(1+x) \quad f'(x) = x(2+3x) \quad f''(x) = 2(1+3x)$$

Ⓐ x-int ( $y=0$ )      Ⓑ y-int ( $x=0$ )

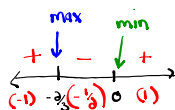
$f(x) = x^2(1+x)$	$f(x) = x^2 + x^3$
$0 = x^2(1+x)$	$f(0) = (0)^2 + (0)^3$
$x^2 = 0$	$f(0) = 0 + 0$
$x = 0$	$f(0) = 0$
$(0, 0)$	$(0, 0)$
$1+x=0$	
$x = -1$	
$(-1, 0)$	

Ⓒ Intervals of Inc/Dec.

$$f'(x) = x(2+3x)$$

$$0 = x(2+3x)$$

$$x = 0 \quad \begin{cases} 2+3x = 0 \\ 3x = -2 \\ x = -\frac{2}{3} \end{cases}$$



Increasing on  $(-\infty, -\frac{2}{3}) \cup (0, \infty)$

Decreasing on  $(-\frac{2}{3}, 0)$

CV:  $x = -\frac{2}{3}, 0$

Ⓓ max @  $x = -\frac{2}{3}$

$$f(x) = x^2 + x^3$$

$$f\left(-\frac{2}{3}\right) = \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^3$$

$$f\left(-\frac{2}{3}\right) = \frac{4}{9} - \frac{8}{27}$$

$$f\left(-\frac{2}{3}\right) = \frac{12}{27} - \frac{8}{27} = \frac{4}{27}$$

$\left(-\frac{2}{3}, \frac{4}{27}\right)$  or  $(-0.6, 0.15)$

Ⓔ min @  $x = 0$

$$f(x) = x^2 + x^3$$

$$f(0) = (0)^2 + (0)^3$$

$$f(0) = 0 + 0 = 0$$

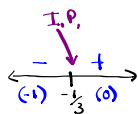
$(0, 0)$

Ⓕ Intervals of Concavity:

$$f''(x) = 2(1+3x)$$

$$0 = 2(1+3x)$$

$$2 \neq 0 \quad \begin{cases} 1+3x = 0 \\ 3x = -1 \\ x = -\frac{1}{3} \end{cases}$$



CD on  $(-\infty, -\frac{1}{3})$

CU on  $(-\frac{1}{3}, \infty)$

CV:  $x = -\frac{1}{3}$

Ⓖ Inflection Point: @  $x = -\frac{1}{3}$

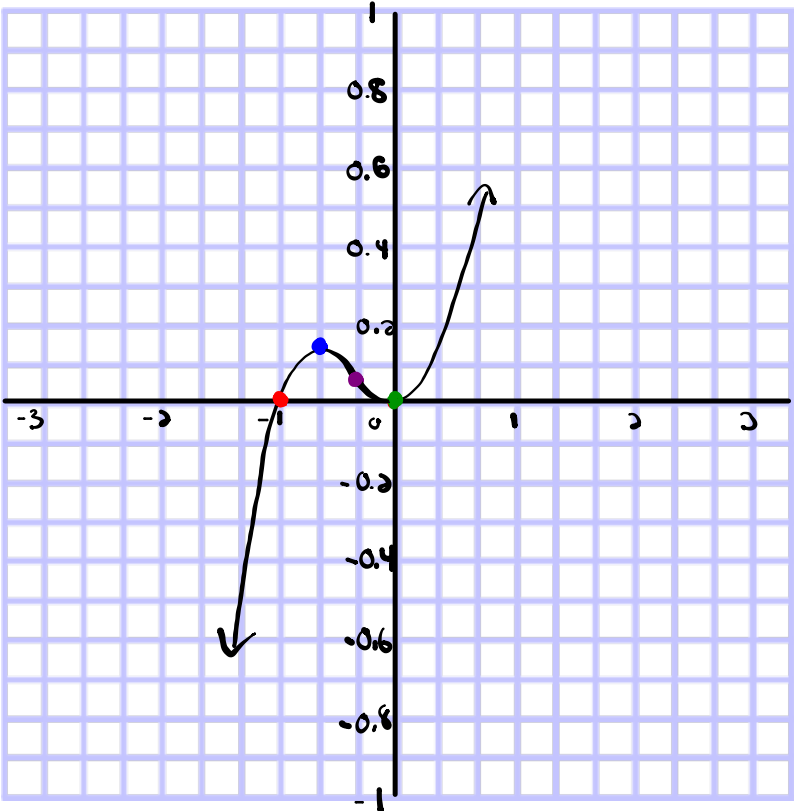
$$f(x) = x^2 + x^3$$

$$f\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^3$$

$$f\left(-\frac{1}{3}\right) = \frac{1}{9} - \frac{1}{27}$$

$$f\left(-\frac{1}{3}\right) = \frac{3}{27} - \frac{1}{27} = \frac{2}{27}$$

$\left(-\frac{1}{3}, \frac{2}{27}\right)$  or  $(-0.3, 0.07)$



homework

Examine the function  $f(x) = \frac{x^2}{1-x^2}$  with respect to...  $f'(x) = \frac{2x}{(1-x^2)^2}$

- Intercepts  $f(x)$
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

$$f''(x) = \frac{2(1+3x^2)}{(1-x^2)^3}$$

ⓐ x-int ( $y=0$ )

$$f(x) = \frac{x^2}{1-x^2}$$

$$0 = \frac{x^2}{1-x^2}$$

$$0 = x^2$$

$$0 = x$$

$$(0,0)$$

ⓑ y-int ( $x=0$ )

$$f(x) = \frac{x^2}{1-x^2}$$

$$f(0) = \frac{0^2}{1-0^2}$$

$$f(0) = \frac{0}{1} = 0$$

$$(0,0)$$

ⓒ Vertical Asymptote: (zeros of the denominator)

$$f(x) = \frac{x^2}{1-x^2}$$

$$\text{VA: } 1-x^2=0$$

$$(1-x)(1+x)=0$$

$$1-x=0 \quad 1+x=0$$

$$1=x \quad x=-1$$

ⓓ Horizontal Asymptote:

$$\lim_{x \rightarrow \infty} \frac{x^2}{1-x^2} = \frac{1}{-1} = -1$$

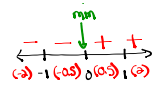
$$y = -1$$

ⓔ Intervals of Inc/Dec.

$$f'(x) = \frac{2x}{(1-x^2)^2}$$

$$\begin{array}{l} 2x=0 \\ x=0 \end{array} \left| \begin{array}{l} (1-x^2)^2=0 \\ 1-x^2=0 \\ 1-x^2 \\ \pm 1=x \end{array} \right.$$

$$\text{Cr: } x = -1, 0, 1$$



Increasing on  $(0, \infty)$   
Decreasing on  $(-\infty, 0)$

ⓕ min @  $x=0$

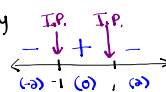
$$f(x) = \frac{x^2}{1-x^2}$$

$$f(0) = \frac{0^2}{1-0^2} = 0$$

$$(0,0)$$

ⓖ Intervals of concavity

$$f''(x) = \frac{2(1+3x^2)}{(1-x^2)^3}$$



$$\begin{array}{l} 2(1+3x^2)=0 \\ 1+3x^2=0 \\ 3x^2=-1 \\ x^2=-\frac{1}{3} \end{array} \left| \begin{array}{l} (1-x^2)^3=0 \\ 1-x^2=0 \\ 1-x^2 \\ \pm 1=x \end{array} \right.$$

Not Possible

(Numerator is always positive)

CO on  $(-\infty, -1) \cup (1, \infty)$   
CU on  $(-1, 1)$

ⓗ Inflection Points:

when  $x=-1$

$$f(x) = \frac{x^2}{1-x^2}$$

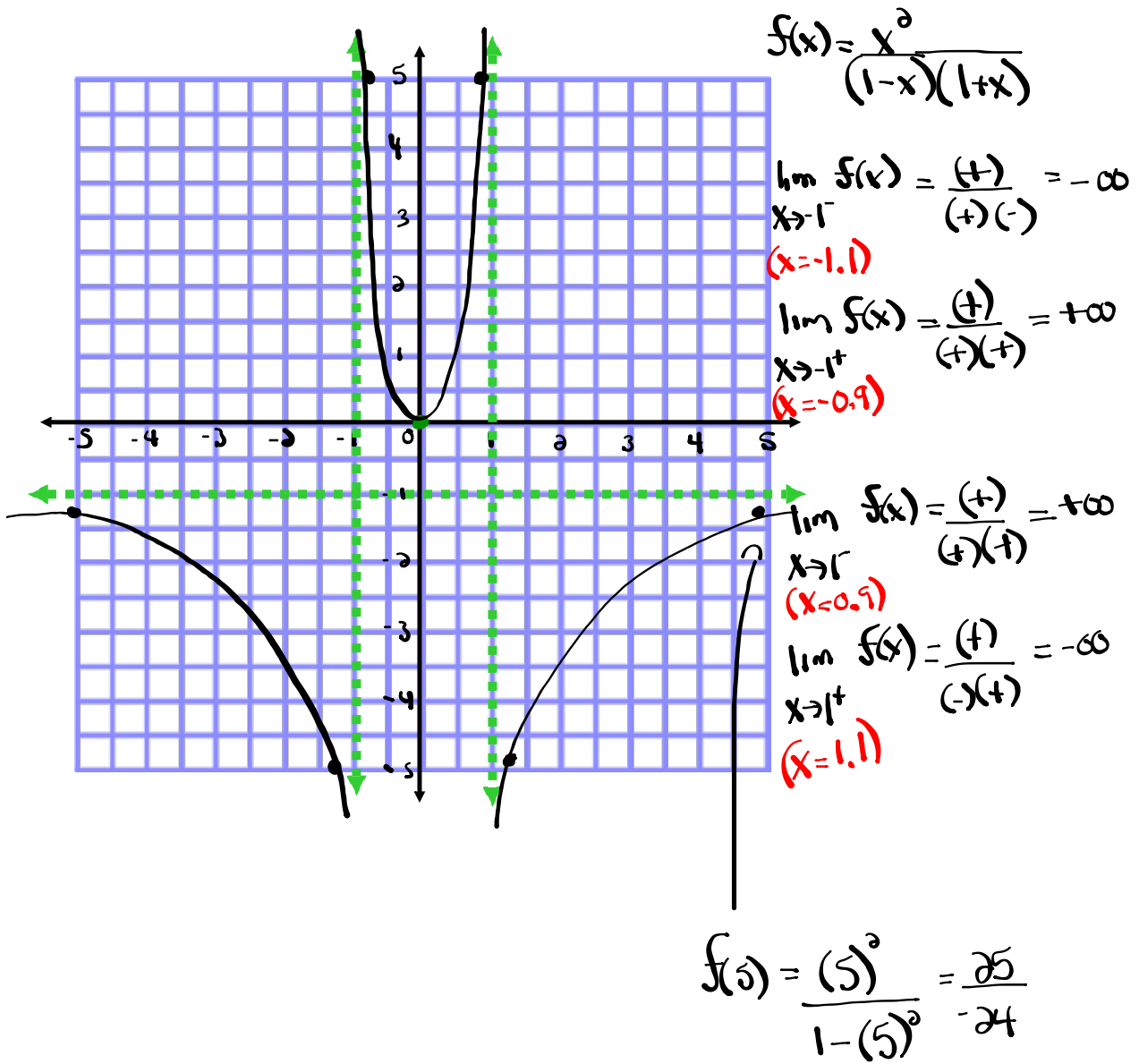
$$f(-1) = \frac{(-1)^2}{1-(-1)^2} = \frac{1}{0} = \text{und.}$$

when  $x=1$

$$f(x) = \frac{x^2}{1-x^2}$$

$$f(1) = \frac{(1)^2}{1-(1)^2} = \frac{1}{0} = \text{und.}$$

\* No Inflection Points  $x = \pm 1$  are the Vertical Asymptotes.



$$f(x) = \frac{x^2}{1-x^2}$$

$$f'(x) = \frac{2x(1-x^2) + 2x(x^2)}{(1-x^2)^2}$$

$$f'(x) = \frac{2x - 2x^3 + 2x^3}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

$$f''(x) = \frac{2(1-x^2)^2 - 2x(2)(1-x^2)(-2x)}{(1-x^2)^4}$$

$$f''(x) = \frac{2(1-x^2)^2 + 8x^2(1-x^2)}{(1-x^2)^4}$$

$$f''(x) = \frac{2(1-x^2)[(1-x^2) + 4x^2]}{(1-x^2)^4} =$$

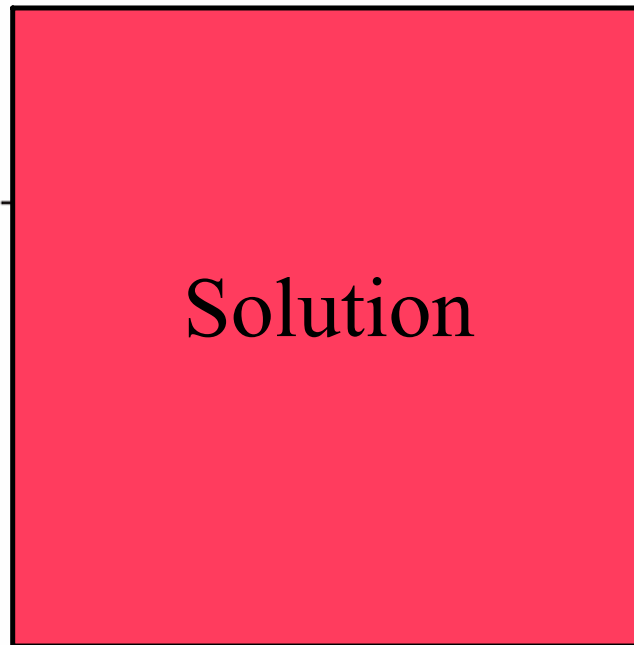
$$f''(x) = \frac{2(1-x^2)(1+3x^2)}{(1-x^2)^4} = \frac{2(1+3x^2)}{(1-x^2)^3}$$



# homework

Examine the function  $f(x) = x^4 - 4x^3$  with respect to...

- Intercepts
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve



homework

Examine the function  $f(x) = \frac{x^2}{x-7}$  with respect to ...

- Intercepts
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

<p>① x-int (y=0)</p> $f(x) = \frac{x^2}{x-7}$ $(x-7) \cdot 0 = \frac{x^2}{x-7} \cdot (x-7)$ $0 = \frac{x^2}{x-7} \cdot (x-7)$ $0 = x^2$ $0 = x$ $(0,0)$	<p>② y-int (x=0)</p> $f(x) = \frac{x^2}{x-7}$ $f(0) = \frac{0^2}{0-7} = \frac{0}{-7} = 0$ $y = 0$ $(0,0)$	<p>③ Symmetry:</p> $f(x) = \frac{x^2}{x-7}$ $f(-x) = \frac{(-x)^2}{(-x)-7}$ $f(-x) = \frac{x^2}{-x-7}$ <p>No symmetry</p>
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④ VA: (denom=0)

$$x-7=0$$

$$x=7$$

$\lim_{x \rightarrow 7^-} \frac{x^2}{x-7} = \frac{49}{0^-} = -\infty$ 
 $\lim_{x \rightarrow 7^+} \frac{x^2}{x-7} = \frac{49}{0^+} = +\infty$

⑤ SA:

$$x-7 \sqrt{\frac{x+7}{x^2}}$$

$$-(x^2-7x)$$

$$-(7x-49)$$

$$49R$$

$y = x+7$ 
 $m = \frac{1}{1}$  rise / run  
 $b = 7$  y-int

⑥ Intervals of Inc/Dec:

$f'(x) = \frac{x(x-14)}{(x-7)^2}$

CV:  $x=0$  |  $x-14=0$  |  $(x-7)^2=0$   
 $x=14$  |  $x=7$

Increasing on  $(-\infty, 0) \cup (14, \infty)$   
 $x < 0$  +  $x > 14$   
 Decreasing on  $(0, 14)$   
 $0 < x < 14$

⑦ Local max/min

$f(x) = \frac{x^2}{x-7}$

When  $x=0$ :  $f(0) = \frac{0^2}{0-7} = \frac{0}{-7} = 0$  local max @  $(0,0)$   
 When  $x=14$ :  $f(14) = \frac{14^2}{14-7} = \frac{196}{7} = 28$  local min @  $(14,28)$

⑧ Intervals of Concavity:

$f''(x) = \frac{98}{(x-7)^3}$

CV:  $98 \neq 0$  |  $(x-7)^3=0$  |  $x-7=0$  |  $x=7$

Concave down on  $(-\infty, 7)$   $x < 7$   
 Concave up on  $(7, \infty)$   $x > 7$

⑨ I.P. (x=7)

$f(x) = \frac{x^2}{x-7}$

$x=7$  is the vertical asymptote.

$f(7) = \frac{7^2}{7-7} = \frac{49}{0} = \text{DNE}$

