Understanding Logarithms

Focus on...

- demonstrating that a logarithmic function is the inverse of an exponential function
- sketching the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- determining the characteristics of the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- explaining the relationship between logarithms and exponents
- expressing a logarithmic function as an exponential function and vice versa
- evaluating logarithms using a variety of methods

exponential

For the exponential function $y = c^x$, the inverse is $x = c^y$. This inverse is also a function and is called a **logarithmic function**. It is written as $y = \log_c x$, where c is a positive number other than 1.

logarithmic

Logarithmic Form **Exponential Form**

exponent

$\log_c x = y$ $c^y = x$

base

Since our number system is based on powers of 10, **logarithms** with base 10 are widely used and are called **common logarithms**. When you write a common logarithm, you do not need to write the base. For example, log 3 means $\log_{10} 3$.

or $\log_{10} 150 = \log 150$

logarithmic function

- a function of the form $y = \log_c x$, where $c > 0$ and $c \neq 1$, that is the inverse of the exponential function $y = c^x$

logarithm

- an exponent
- in $x = c^y$, y is called the logarithm to base c of x

common logarithm

- a logarithm with base 10

Write each of the following in logarithmic form

a) ans exp

$32 = 2^5$

Base

$\log_2(32) = 5$

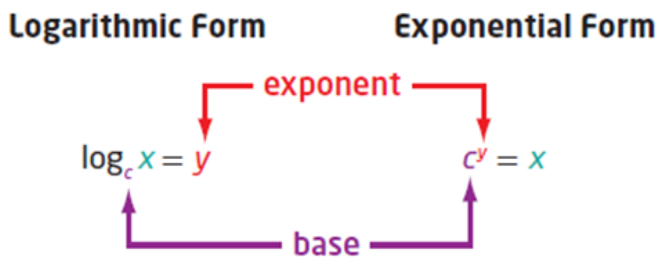
b) $2^{-5} = \frac{1}{32}$

$\log_2\left(\frac{1}{32}\right) = -5$

c) $x = 10^y$

$\log_{10}(x) = y$

$\log x = y$



Write each of the following in exponential form

a) ans

$\log_4 16 = 2$

Base exp

b) $\log_2\left(\frac{1}{32}\right) = -5$

c) $\log 65 = 1.8129$

$4^2 = 16$

$2^{-5} = \frac{1}{32}$

$10^{1.8129} = 65$

Example 1

Evaluating a Logarithm

Evaluate.

a) $\log_7 49 = 2$ **b)** $\log_6 1 = 0$ **c)** $\log 0.001 = -3$ **d)** $\log_2 \sqrt{8} = 1.5$

a)

$\frac{\log 49}{\log 7} = 2$

$x = \log_7 49$

$7^x = 49$

$\cancel{7}^x = (\cancel{7})^2$

$x = 2$

$\boxed{\log_7 49 = 2}$

b)

$\frac{\log 1}{\log 6} = 0$

$x = \log_6 1$

$6^x = 1$

$\cancel{6}^x = (\cancel{6})^0$

$x = 0$

$\boxed{\log_6 1 = 0}$

c)

$\frac{\log 0.001}{\log 10} = -3$

$x = \log 0.001$

$10^x = 0.001$

$\cancel{10}^x = (\cancel{10})^{-3}$

$x = -3$

$\boxed{\log 0.001 = -3}$

d)

$\frac{\log \sqrt{8}}{\log 2} = 1.5$

$x = \log_2 \sqrt{8}$

$2^x = \sqrt{8}$

$2^x = (8)^{1/2}$

$2^x = (2^3)^{1/2}$

$\cancel{2}^x = \cancel{2}^{3/2}$

$x = \frac{3}{2}$

$\boxed{\log_2 \sqrt{8} = \frac{3}{2}}$

Example 2

Determine an Unknown in an Expression in Logarithmic Form

Determine the value of x. (Convert to exponential form)

a) $\log_5 x = -3$

b) $\log_x 36 = 2$

c) $\log_{64} x = \frac{2}{3}$

a) $\log_5 x = -3$ (log. form)

$5^{-3} = x$ (exp. form)

$\left(\frac{1}{5}\right)^3 = x$

$\boxed{\frac{1}{125} = x}$

b) $\log_x 36 = 2$ (log. form)

$x^2 = 36$ (exp. form)

$x = \pm 6$

$\boxed{x = 6}$ $c > 0$

c) $\log_{64} x = \frac{2}{3}$ (log. form)

$64^{2/3} = x$ (exp. form)

$16 = x$

exp.

$2^4 = 16$ (exponential form)

Base ans.

$\log_2(16) = 4$ (logarithmic form)

Chapter 7	Inverse	Chapter 8
Exponential		Logarithmic
$y = c^x$, $c > 0, c \neq 1$		$y = \log_c x$, $c > 0, c \neq 1$
D: $\{x \mid x \in R\}$		D: $\{x \mid x > 0, x \in R\}$ or $(0, \infty)$
R: $\{y \mid y > 0, y \in R\}$		R: $\{y \mid y \in R\}$ or $(-\infty, \infty)$
x-int: none		x-int: $x = 1$ $(1, 0)$
y-int: $y = 1$ or $(0, 1)$		y-int: none
HA: $y = 0$		VA: $x = 0$

Example 3

Graph the Inverse of an Exponential Function

a) State the inverse of $f(x) = 3^x$.

b) Sketch the graph of the inverse. Identify the following characteristics of the inverse graph: $(x,y) \to (y,x)$

- the domain and range
- the x-intercept, if it exists
- the y-intercept, if it exists
- the equations of any asymptotes

$y = 3^x$ passes the HLT

a) $f(x) = 3^x$

$y = 3^x$

$x = 3^y$ (exp. form)

$\log_3 x = y$ (log form)

$y = \log_3 x$

$\boxed{f^{-1}(x) = \log_3 x}$

b) $y = 3^x$ $\quad (x,y) \to (y,x)$ $\quad$ Inverse $y = \log_3 x$

x	y
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9

x	y
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2

Solution

a) The inverse of $y = f(x) = 3^x$ is $x = 3^y$ or, expressed in logarithmic form, $y = \log_3 x$. Since the inverse is a function, it can be written in function notation as

How do you know that $y = \log_3 x$ is a function?

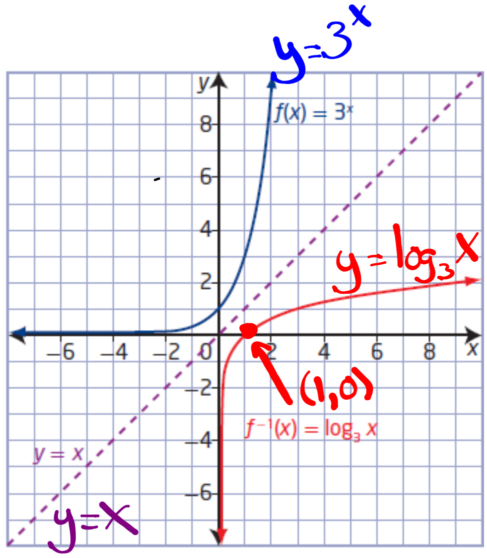
b) Set up tables of values for both the exponential function, $f(x)$, and its inverse, $f^{-1}(x)$. Plot the points and join them with a smooth curve.

$y = 3^x$

$f(x) = 3^x$	
x	y
-3	$\frac{1}{27}$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9
3	27

$y = \log_3 x$

$f^{-1}(x) = \log_3 x$	
x	y
$\frac{1}{27}$	-3
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2
27	3



The graph of the inverse, $f^{-1}(x) = \log_3 x$, is a reflection of the graph of $f(x) = 3^x$ about the line $y = x$. For $\boxed{f^{-1}(x) = \log_3 x}$,

- the domain is $\{x \mid x > 0, x \in R\}$ $(0, \infty)$ and the range is $\{y \mid y \in R\}$ or $(-\infty, \infty)$
- the x-intercept is 1 or $(1, 0)$
- there is no y-intercept
- the vertical asymptote, the y-axis, has equation $x = 0$; there is no horizontal asymptote

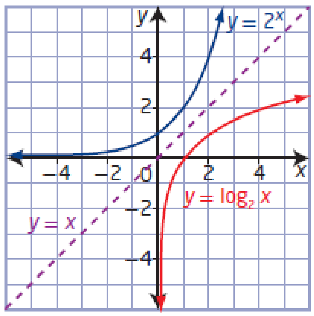
How do the characteristics of $f^{-1}(x) = \log_3 x$ compare to the characteristics of $f(x) = 3^x$?

Key Ideas

- A logarithm is an exponent.
- Equations in exponential form can be written in logarithmic form and vice versa.

Exponential Form	**Logarithmic Form**
$x = c^y$	$y = \log_c x$

- The inverse of the exponential function $y = c^x$, $c > 0$, $c \neq 1$, is $x = c^y$ or, in logarithmic form, $y = \log_c x$. Conversely, the inverse of the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$, is $x = \log_c y$ or, in exponential form, $y = c^x$.
- The graphs of an exponential function and its inverse logarithmic function are reflections of each other in the line $y = x$, as shown.



- For the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$,
  - the domain is $\{x \mid x > 0, x \in R\}$
  - the range is $\{y \mid y \in R\}$
  - the x-intercept is 1
  - the vertical asymptote is $x = 0$, or the y-axis
- A common logarithm has base 10. It is not necessary to write the base for common logarithms:

  $\log_{10} x = \log x$

Questions from Homework

① a) $y=2^x$

$x=2^y$

$\log_2(x)=y$

$y=\log_2 x$

D: $\{x|x>0, x\in R$

R: $\{y|y\in R\}$

x-int: $x=1$

y-int: none

VA: $x=0$

b) $y=\left(\frac{1}{3}\right)^x$

$x=\left(\frac{1}{3}\right)^y$

$\log_{1/3}(x)=y$

$y=\log_{1/3}x$

D: $\{x|x>0, x\in R$

R: $\{y|y\in R\}$

x-int: $x=1$

y-int: none

VA: $x=0$

17. The growth of a new social networking site can be modelled by the exponential function $N(t) = 1.1^t$, where N is the number of users after t days.

a) Write the equation of the inverse.

b) How long will it take, to the nearest day, for the number of users to exceed 1 000 000? $N = 1000000$

$t = ?$

a) $N=1.1^t$

$t=1.1^N$ (Exp form)

$\log_{1.1}t=N$

$\boxed{N=\log_{1.1}t}$

b) $N=1.1^t$

$1000000=1.1^t$

$\cancel{(1.1)}^{145} = \cancel{1.1}^t$

$\boxed{145=t}$

② d) $7^{2x} = \underbrace{y+3}_{\text{ans}} \longrightarrow \log_7(y+3) = 2x$

exp

Base

③ c) $\log_{10}(1\,000\,000) = 6 \longrightarrow 10^6 = 1\,000\,000$

Base ans exp.

④ c) $\log_4 \sqrt[3]{4} = \frac{1}{3}$

$$\frac{\log(4^{1/3})}{\log 4} = \frac{1}{3}$$

$x = \log_4 (4)^{1/3}$ (log form)

$\cancel{4}^x = \cancel{4}^{1/3}$ (exp. form)

$x = 1/3$

⑤ $a < \log_2 28 < b$

$4 < \log_2 28 < 5$

$2^4 = 16$

$2^5 = 32$

$\log_2 28 = 4.8$

$$\frac{\log 28}{\log 2} = 4.8$$

⑫ c) $\log_{1/4} x = -3 \longrightarrow \left(\frac{1}{4}\right)^{-3} = x$

Base ans exp

$4^3 = x$

$\boxed{64 = x}$

⑬ a) 5^m where $m = \log_5 7$

$m = \frac{\log 7}{\log 5} = \underline{1.2091}$

Solve: 5^m

$= 5^{1.2091}$

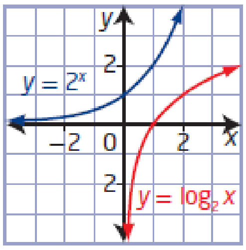
$= 7$

$5^{\log_5 7} = 7$

$2^{\log_2 16}$

8.1 Understanding Logarithms, pages 380 to 382

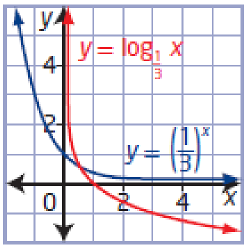
1. a) i)



ii) $y = \log_2 x$

iii) domain $\{x \mid x > 0, x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$, x-intercept 1, no y-intercept, vertical asymptote $x = 0$

b) i)



ii) $y = \log_{\frac{1}{3}} x$

iii) domain $\{x \mid x > 0, x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$, x-intercept 1, no y-intercept, vertical asymptote $x = 0$

2. a) $\log_{12} 144 = 2$ **b)** $\log_8 2 = \frac{1}{3}$

c) $\log_{10} 0.000\,01 = -5$ **d)** $\log_7 (y + 3) = 2x$

3. a) $5^2 = 25$ **b)** $8^{\frac{2}{3}} = 4$

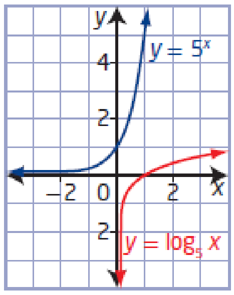
c) $10^6 = 1\,000\,000$ **d)** $11^y = x + 3$

4. a) 3 **b)** 0 **c)** $\frac{1}{3}$ **d)** -3

5. $a = 4$; $b = 5$

8. a) $y = \log_5 x$

b)



domain $\{x \mid x > 0, x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$, x-intercept 1, no y-intercept, vertical asymptote $x = 0$

10. They are reflections of each other in the line $y = x$.

11. a) They have the exact same shape.

b) One of them is increasing and the other is decreasing.

12. a) 216 **b)** 81 **c)** 64 **d)** 8

13. a) 7 **b)** 6

14. a) 0 **b)** 1

15. -1

16. 16

17. a) $t = \log_{1.1} N$ **b)** 145 days

18. The larger asteroid had a relative risk that was 1479 times as dangerous.

19. 1000 times as great

20. 5

21. $m = 14$, $n = 13$

22. $4n$

23. $y = 3^{2^x}$

Transformations of Logarithmic Functions

$y = a\log_c(b(x-h)) + k$

Focus on...

- explaining the effects of the parameters a, b, h, and k in $y = a\log_c(b(x - h)) + k$ on the graph of $y = \log_c x$, where $c > 1$
- sketching the graph of a logarithmic function by applying a set of transformations to the graph of $y = \log_c x$, where $c > 1$, and stating the characteristics of the graph

Remember:

Parameter	Transformation
a	$(x, y) \rightarrow (x, ay)$
b	$(x, y) \rightarrow \left(\frac{x}{b}, y\right)$
h	$(x, y) \rightarrow (x + h, y)$
k	$(x, y) \rightarrow (x, y + k)$

Example 1

Translations of a Logarithmic Function

a) Use transformations to sketch the graph of the function $y = \log_3 (x + 9) + 2$.

b) Identify the following characteristics of the graph of the function.

- **i)** the equation of the asymptote
- **ii)** the domain and range
- **iii)** the y-intercept, if it exists
- **iv)** the x-intercept, if it exists

a) $y = 1\log_3 1(x+9)+2$ $c = 3 \rightarrow$ base

$a = 1 \rightarrow$ no vertical stretch or reflection

$b = 1 \rightarrow$ no horizontal stretch or reflection

$h = -9 \rightarrow$ 9 units left

$k = 2 \rightarrow$ 2 units up

$(x,y) \longrightarrow \left(\frac{1}{1}x + (-9),\ 1y + 2\right)$

$(x,y) \rightarrow (x-9,\ y+2)$

$y = 3^x$

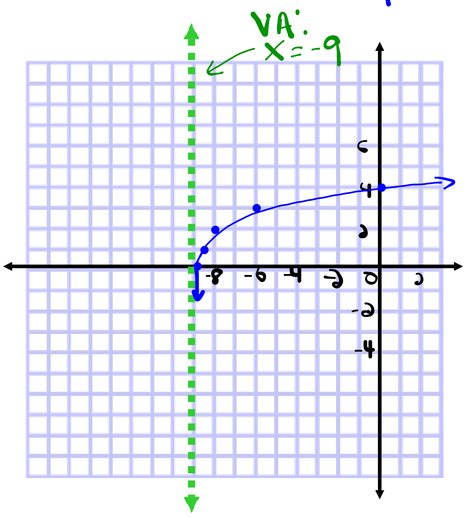
x	y
-2	1/9
-1	1/3
0	1
1	3
2	9

$y = \log_3 x$

x	y
1/9	-2
1/3	-1
1	0
3	1
9	2

$(x,y) \rightarrow (x-9,\ y+2)$

x	y
$-8.\overline{8}$ or $-80/9$	0
$-8.\overline{6}$ or $-26/3$	1
-8	2
-6	3
0	4



b) (i) VA: $x = -9$

(ii) D: $\{x \mid x > -9, x \in R\}$ or $(-9, \infty)$

R: $\{y \mid y \in R\}$

(iii) y int $(x=0)$

$y = \log_3(x+9)+2$

$y = \log_3(0+9)+2$

$y = \boxed{\log_3(9)}+2$

$\frac{\log 9}{\log 3} = 2$

$y = 2 + 2$

$y = 4$

$(0,4)$

(iv) x int $(y=0)$

$y = \log_3(x+9)+2$

$0 = \log_3(x+9)+2$

$-2 = \log_3(x+9)$ (log form)

exp Base ans

$3^{-2} = x+9$

$\left(\frac{1}{3}\right)^2 = x+9$

$\frac{1}{9} = x+9$

$\frac{1}{9} - \frac{9}{1} = x$

$\frac{1}{9} - \frac{81}{9} = x$

$-\frac{80}{9} = x$

$(-8.\overline{8}, 0)$

Example 2

Reflections, Stretches, and Translations of a Logarithmic Function

a) Use transformations to sketch the graph of the function $y = -\log_2(2x + 6)$. $\rightarrow$ $y = -\log_2(2(x+3))$

b) Identify the following characteristics of the graph of the function.

i) the equation of the asymptote
ii) the domain and range
iii) the y-intercept, if it exists
iv) the x-intercept, if it exists

a) $a = -1 \rightarrow$ reflected in the x-axis, no vertical stretch

$b = 2 \rightarrow$ no reflection in y-axis and horizontally compressed by a factor of $\frac{1}{2}$

$h = -3 \rightarrow$ horizontal translation 3 units left

$k = 0 \rightarrow$ no vertical translation.

$y = 2^x$

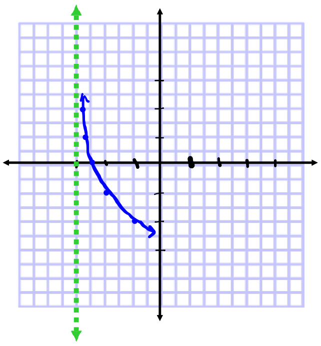
x	y
-2	1/4
-1	1/2
0	1
1	2
2	4

$y = \log_2 x$

x	y
1/4	-2
1/2	-1
1	0
2	1
4	2

$(x, y) \rightarrow \left(\frac{1}{2}x - 3, -1y\right)$

x	y
(-2.88) $-2\frac{7}{8}$	2
(-2.75) $-1\frac{1}{4}$	1
(-2.5) $-\frac{5}{2}$	0
-2	-1
-1	-2



b) (i) VA: $x = -3$

(ii) D: $\{x \mid x > -3, x \in R\}$

R: $\{y \mid y \in R\}$

(iii) y-int $(x = 0)$

$y = -\log_2(2(x+3))$

$y = -\log_2(2(0+3))$

$y = -\boxed{\log_2(6)}$ $\quad \frac{\log 6}{\log 2} = 2.58$

$y = -(2.58)$

$y = -2.58$

$(0, -2.58)$

(iv) x-int $(y = 0)$

$y = -\log_2(2(x+3))$

$\frac{0}{-1} = \frac{-1\log_2(2(x+3))}{-1}$ divide by "a"

$0 = \log_2(2(x+3))$ (log)

$2^0 = 2(x+3)$ (exp)

$\frac{1}{2} = \frac{2(x+3)}{2}$

$\frac{1}{2} = x + 3$

$\frac{1}{2} - \frac{3}{1} = x$

$\frac{1}{2} - \frac{6}{2} = x$

$-\frac{5}{2} = x$

$(-2.5, 0)$

Key Ideas

- To represent real-life situations, you may need to transform the basic logarithmic function $y = \log_b x$ by applying reflections, stretches, and translations. These transformations should be performed in the same manner as those applied to any other function.
- The effects of the parameters a, b, h, and k in $y = a \log_c (b(x - h)) + k$ on the graph of the logarithmic function $y = \log_c x$ are shown below.

Vertically stretch by a factor of $|a|$ about the x-axis. Reflect in the x-axis if $a < 0$.

Vertically translate k units.

$$y = a \log_c (b(x - h)) + k$$

Horizontally stretch by a factor of $\left|\frac{1}{b}\right|$ about the y-axis. Reflect in the y-axis if $b < 0$.

Horizontally translate h units.

- Only parameter h changes the vertical asymptote and the domain. None of the parameters change the range.

Homework

Questions #1, 2, 4, 5, 8, 11 on page 389 - 391