

# Understanding Logarithms

## Focus on...

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- demonstrating that a logarithmic function is the inverse of an exponential function
- sketching the graph of  $y = \log_c x, c > 0, c \neq 1$
- determining the characteristics of the graph of  $y = \log_c x, c > 0, c \neq 1$
- explaining the relationship between logarithms and exponents
- expressing a logarithmic function as an exponential function and vice versa
- evaluating logarithms using a variety of methods

*exponential*

For the exponential function  $y = c^x$ , the inverse is  $x = c^y$ . This inverse is also a function and is called a **logarithmic function**. It is written as  $y = \log_c x$ , where  $c$  is a positive number other than 1.

Logarithmic

**Logarithmic Form**

**Exponential Form**



Since our number system is based on powers of 10, **logarithms** with base 10 are widely used and are called **common logarithms**. When you write a common logarithm, you do not need to write the base. For example,  $\log 3$  means  $\log_{10} 3$ .

or  $\log_{10} 150 = \log 150$

**logarithmic function**

- a function of the form  $y = \log_c x$ , where  $c > 0$  and  $c \neq 1$ , that is the inverse of the exponential function  $y = c^x$

**logarithm**

- an exponent
- in  $x = c^y$ ,  $y$  is called the logarithm to base  $c$  of  $x$

**common logarithm**

- a logarithm with base 10

Write each of the following in logarithmic form

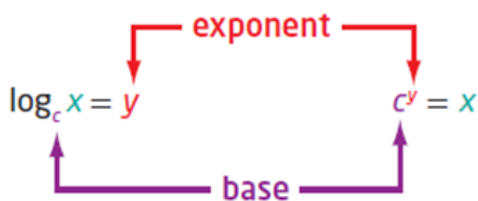
a)  $32 = 2^5$   
 (Handwritten: "ans" points to 5, "exp" points to 2, "Base" points to 2)  
 $\log_2(32) = 5$

b)  $2^{-5} = \frac{1}{32}$   
 $\log_2\left(\frac{1}{32}\right) = -5$

c)  $x = 10^y$   
 $\log_{10}(x) = y$   
 $\log x = y$

Logarithmic Form

Exponential Form



Write each of the following in exponential form

a)  $\log_4 16 = 2$   
 (Handwritten: "ans" points to 2, "exp" points to 4, "Base" points to 4)  
 $4^2 = 16$

b)  $\log_2\left(\frac{1}{32}\right) = -5$   
 $2^{-5} = \frac{1}{32}$

c)  $\log 65 = 1.8129$   
 $10^{1.8129} = 65$

## Example 1

### Evaluating a Logarithm

Evaluate.

a)  $\log_7 49 = ?$

$$\frac{\log 49}{\log 7} = ?$$

$$x = \log_7 49$$

$$7^x = 49$$

~~$$7^x = (7)^2$$~~

$$x = 2$$

$$\boxed{\log_7 49 = 2}$$

b)  $\log_6 1 = 0$

$$\frac{\log 1}{\log 6} = 0$$

$$x = \log_6 1$$

$$6^x = 1$$

~~$$6^x = (6)^0$$~~

$$x = 0$$

$$\boxed{\log_6 1 = 0}$$

c)  $\log 0.001 = -3$

$$\frac{\log 0.001}{\log 10} = -3$$

$$x = \log 0.001$$

$$10^x = 0.001$$

~~$$10^x = (10)^{-3}$$~~

$$x = -3$$

$$\boxed{\log 0.001 = -3}$$

d)  $\log_2 \sqrt{8} = 1.5$

$$\frac{\log \sqrt{8}}{\log 2} = 1.5$$

$$x = \log_2 \sqrt{8}$$

$$2^x = \sqrt{8}$$

$$2^x = (8)^{1/2}$$

$$2^x = (2^3)^{1/2}$$

~~$$2^x = 2^{3/2}$$~~

$$x = \frac{3}{2}$$

$$\boxed{\log_2 \sqrt{8} = \frac{3}{2}}$$

## Example 2

### Determine an Unknown in an Expression in Logarithmic Form

Determine the value of  $x$ . (convert to exponential form)

a)  $\log_5 x = -3$

b)  $\log_x 36 = 2$

c)  $\log_{64} x = \frac{2}{3}$

a)  $\log_5 x = -3$  (log. form)

$$5^{-3} = x \text{ (exp. form)}$$

$$\left(\frac{1}{5}\right)^3 = x$$

$$\boxed{\frac{1}{125} = x}$$

b)  $\log_x 36 = 2$  (log. form)

$$x^2 = 36 \text{ (exp. form)}$$

$$x = \pm 6$$

$$\boxed{x = 6} \quad c > 0$$

c)  $\log_{64} x = \frac{2}{3}$  (log. form)

$$64^{\frac{2}{3}} = x \text{ (exp. form)}$$

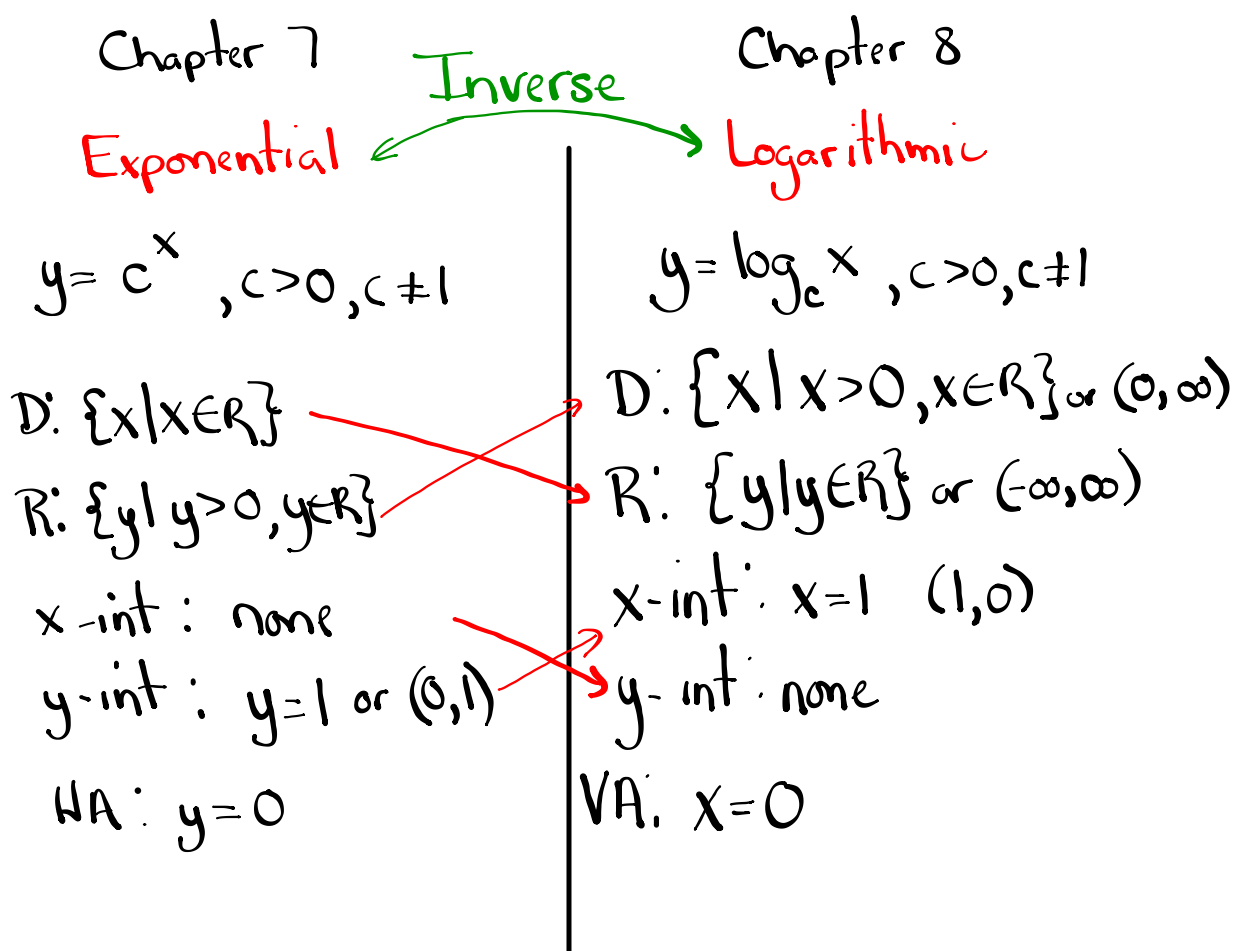
$$16 = x$$

$$2^4 = 16 \quad (\text{exponential form})$$

↑ Base                      ↑ ans.

← exp.

$$\log_2(16) = 4 \quad (\text{logarithmic form})$$



### Example 3

#### Graph the Inverse of an Exponential Function

- a) State the inverse of  $f(x) = 3^x$ .
- b) Sketch the graph of the inverse. Identify the following characteristics of the inverse graph:  $(x, y) \rightarrow (y, x)$
- the domain and range
  - the x-intercept, if it exists
  - the y-intercept, if it exists
  - the equations of any asymptotes

$$a) f(x) = 3^x$$

$$y = 3^x$$

$$x = 3^y \quad (\text{exp. form})$$

$$\log_3 x = y \quad (\text{log form})$$

$$y = \log_3 x$$

$$f^{-1}(x) = \log_3 x$$

$y = 3^x$  passes the HLT

$$b) y = 3^x \quad (x, y) \rightarrow (y, x)$$

x	y
-2	1/9
-1	1/3
0	1
1	3
2	9

Inverse

$$y = \log_3 x$$

x	y
1/9	-2
1/3	-1
1	0
3	1
9	2



**Solution**

a) The inverse of  $y = f(x) = 3^x$  is  $x = 3^y$  or, expressed in logarithmic form,  $y = \log_3 x$ . Since the inverse is a function, it can be written in function notation as

How do you know that  $y = \log_3 x$  is a function?

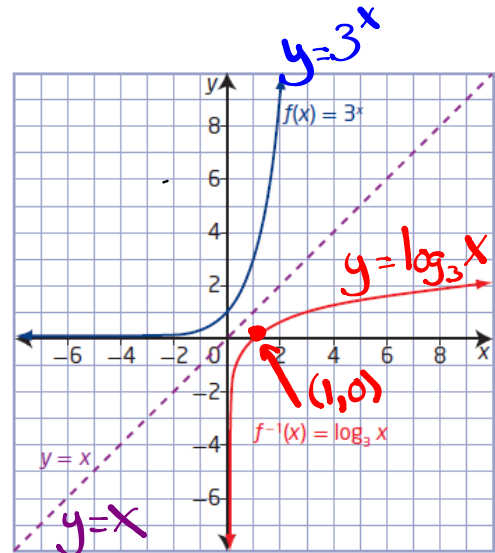
b) Set up tables of values for both the exponential function,  $f(x)$ , and its inverse,  $f^{-1}(x)$ . Plot the points and join them with a smooth curve.

$y = 3^x$   
 $f(x) = 3^x$

x	y
-3	$\frac{1}{27}$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9
3	27

$y = \log_3 x$   
 $f^{-1}(x) = \log_3 x$

x	y
$\frac{1}{27}$	-3
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2
27	3



The graph of the inverse,  $f^{-1}(x) = \log_3 x$ , is a reflection of the graph

of  $f(x) = 3^x$  about the line  $y = x$ . For  $f^{-1}(x) = \log_3 x$ ,

- the domain is  $\{x \mid x > 0, x \in \mathbb{R}\}$  and the range is  $\{y \mid y \in \mathbb{R}\}$  or  $(-\infty, \infty)$
- the x-intercept is 1 or  $(1, 0)$
- there is no y-intercept
- the vertical asymptote, the y-axis, has equation  $x = 0$ ; there is no horizontal asymptote

How do the characteristics of  $f^{-1}(x) = \log_3 x$  compare to the characteristics of  $f(x) = 3^x$ ?

### Key Ideas

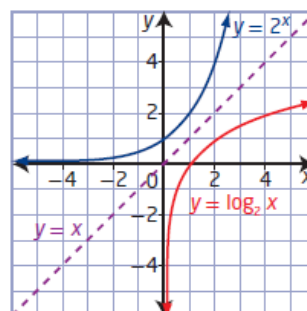
- A logarithm is an exponent.
- Equations in exponential form can be written in logarithmic form and vice versa.

**Exponential Form      Logarithmic Form**

$$x = c^y \qquad y = \log_c x$$

- The inverse of the exponential function  $y = c^x$ ,  $c > 0$ ,  $c \neq 1$ , is  $x = c^y$  or, in logarithmic form,  $y = \log_c x$ . Conversely, the inverse of the logarithmic function  $y = \log_c x$ ,  $c > 0$ ,  $c \neq 1$ , is  $x = \log_c y$  or, in exponential form,  $y = c^x$ .
- The graphs of an exponential function and its inverse logarithmic function are reflections of each other in the line  $y = x$ , as shown.
- For the logarithmic function  $y = \log_c x$ ,  $c > 0$ ,  $c \neq 1$ ,
  - the domain is  $\{x \mid x > 0, x \in \mathbb{R}\}$
  - the range is  $\{y \mid y \in \mathbb{R}\}$
  - the x-intercept is 1
  - the vertical asymptote is  $x = 0$ , or the y-axis
- A common logarithm has base 10. It is not necessary to write the base for common logarithms:

$$\log_{10} x = \log x$$



## Questions from Homework

$$\textcircled{1} \text{ a) } y = 2^x$$

$$x = 2^y$$

$$\log_2(x) = y$$

$$y = \log_2 x$$

$$D: \{x \mid x > 0, x \in \mathbb{R}\}$$

$$R: \{y \mid y \in \mathbb{R}\}$$

$$x\text{-int: } x = 1$$

$$y\text{-int: none}$$

$$VA: x = 0$$

$$\text{b) } y = \left(\frac{1}{3}\right)^x$$

$$x = \left(\frac{1}{3}\right)^y$$

$$\log_{\frac{1}{3}}(x) = y$$

$$y = \log_{\frac{1}{3}} x$$

$$D: \{x \mid x > 0, x \in \mathbb{R}\}$$

$$R: \{y \mid y \in \mathbb{R}\}$$

$$x\text{-int: } x = 1$$

$$y\text{-int: none}$$

$$VA: x = 0$$

17. The growth of a new social networking site can be modelled by the exponential function  $N(t) = 1.1^t$ , where  $N$  is the number of users after  $t$  days.

- Write the equation of the inverse.
- How long will it take, to the nearest day, for the number of users to exceed 1 000 000?

$$\textcircled{2} \text{ d) } 7^{2x} = y+3 \rightarrow \log_7(y+3) = 2x$$

↑ Base      ↑ ans      ← exp

$$\textcircled{3} \text{ c) } \log_{10}(1000000) = 6 \rightarrow 10^6 = 1000000$$

↑ Base      ↑ ans      ↑ exp.

$$\textcircled{4} \text{ c) } \log_4 \sqrt[3]{4} = \frac{1}{3}$$

$$\frac{\log(4^{1/3})}{\log 4} = \frac{1}{3}$$

$$x = \log_4(4)^{1/3} \text{ (log form)}$$

~~$$4^x = 4^{1/3} \text{ (exp. form)}$$~~

$$x = \frac{1}{3}$$

$$\textcircled{5} \quad a < \log_p 28 < b \quad 2^4 = 16$$

↑

$$4 < \log_p 28 < 5 \quad 2^5 = 32$$

$$\log_p 28 = 4.8$$

$$\frac{\log 28}{\log p} = 4.8$$

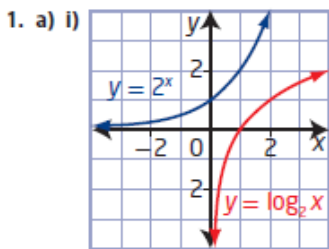
$$\textcircled{1a} \text{ c) } \log_{1/4} x = -3 \rightarrow \left(\frac{1}{4}\right)^{-3} = x$$

↑ Base      ↑ ans      ↑ exp

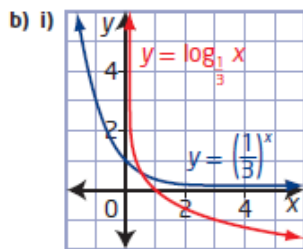
$$4^3 = x$$

$64 = x$

8.1 Understanding Logarithms, pages 380 to 382

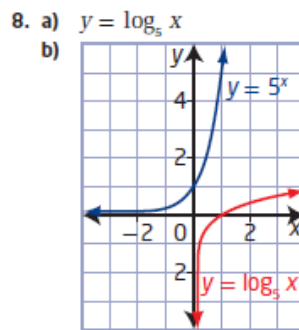


ii)  $y = \log_2 x$   
 iii) domain  $\{x \mid x > 0, x \in \mathbb{R}\}$ ,  
 range  $\{y \mid y \in \mathbb{R}\}$ ,  
 x-intercept 1, no y-intercept,  
 vertical asymptote  $x = 0$



ii)  $y = \log_{\frac{1}{3}} x$   
 iii) domain  $\{x \mid x > 0, x \in \mathbb{R}\}$ ,  
 range  $\{y \mid y \in \mathbb{R}\}$ ,  
 x-intercept 1, no y-intercept,  
 vertical asymptote  $x = 0$

2. a)  $\log_{12} 144 = 2$       b)  $\log_8 2 = \frac{1}{3}$   
 c)  $\log_{10} 0.000\ 01 = -5$       d)  $\log_7 (y + 3) = 2x$   
 3. a)  $5^2 = 25$       b)  $8^{\frac{2}{3}} = 4$   
 c)  $10^6 = 1\ 000\ 000$       d)  $11^y = x + 3$   
 4. a) 3      b) 0      c)  $\frac{1}{3}$       d) -3  
 5.  $a = 4; b = 5$



domain  $\{x \mid x > 0, x \in \mathbb{R}\}$ ,  
 range  $\{y \mid y \in \mathbb{R}\}$ ,  
 x-intercept 1,  
 no y-intercept,  
 vertical asymptote  $x = 0$

10. They are reflections of each other in the line  $y = x$ .  
 11. a) They have the exact same shape.  
 b) One of them is increasing and the other is decreasing.  
 12. a) 216      b) 81      c) 64      d) 8  
 13. a) 7      b) 6  
 14. a) 0      b) 1  
 15. -1  
 16. 16  
 17. a)  $t = \log_{1.1} N$       b) 145 days  
 18. The larger asteroid had a relative risk that was 1479 times as dangerous.  
 19. 1000 times as great  
 20. 5  
 21.  $m = 14, n = 13$   
 22.  $4n$   
 23.  $y = 3^{2^x}$

# Transformations of Logarithmic Functions

$$y = a \log_c (b(x-h)) + k$$

Focus on...

- explaining the effects of the parameters  $a$ ,  $b$ ,  $h$ , and  $k$  in  $y = a \log_c (b(x-h)) + k$  on the graph of  $y = \log_c x$ , where  $c > 1$
- sketching the graph of a logarithmic function by applying a set of transformations to the graph of  $y = \log_c x$ , where  $c > 1$ , and stating the characteristics of the graph

## Remember:

Parameter	Transformation
$a$	$(x, y) \rightarrow (x, ay)$
$b$	$(x, y) \rightarrow \left(\frac{x}{b}, y\right)$
$h$	$(x, y) \rightarrow (x + h, y)$
$k$	$(x, y) \rightarrow (x, y + k)$

Example 1

Translations of a Logarithmic Function

- a) Use transformations to sketch the graph of the function  
 $y = \log_3(x + 9) + 2$ .
- b) Identify the following characteristics of the graph of the function.
- i) the equation of the asymptote
  - ii) the domain and range
  - iii) the y-intercept, if it exists
  - iv) the x-intercept, if it exists

a)  $y = 1 \log_3(x+9) + 2$      $c = 3 \rightarrow$  base

$a = 1 \rightarrow$  no vertical stretch or reflection

$b = 1 \rightarrow$  no horizontal stretch or reflection

$h = -9 \rightarrow$  9 units left

$k = 2 \rightarrow$  2 units up

$(x, y) \rightarrow \left(\frac{1}{1}x + (-9), 1y + 2\right)$

$(x, y) \rightarrow (x - 9, y + 2)$

$y = 3^x$

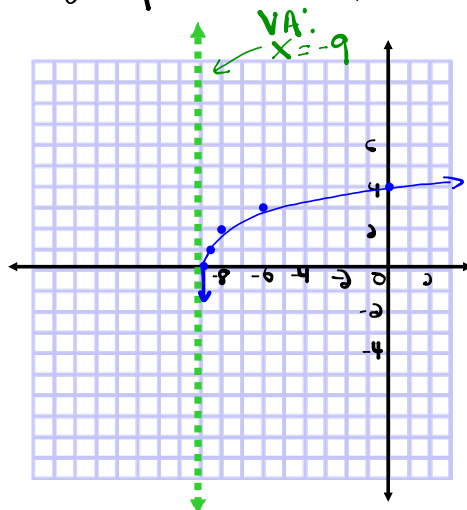
x	y
-2	1/9
-1	1/3
0	1
1	3
2	9

$y = \log_3 x$

x	y
1/9	-2
1/3	-1
1	0
3	1
9	2

$(x, y) \rightarrow (x - 9, y + 2)$

x	y
-8.8 or -80/9	0
-8.6 or -26/3	1
-8	2
-6	3
0	4



b) VA:  $x = -9$

(i) D:  $\{x \mid x > -9, x \in \mathbb{R}\}$  or  $(-9, \infty)$

R:  $\{y \mid y \in \mathbb{R}\}$

(iii) y int ( $x=0$ )

$$y = \log_3(x+9) + 2$$

$$y = \log_3(0+9) + 2$$

$$y = \log_3(9) + 2$$

$$y = 2 + 2$$

$$y = 4$$

$$(0, 4)$$

$$\frac{\log 9}{\log 3} = 2$$

(iv) x int ( $y=0$ )

$$y = \log_3(x+9) + 2$$

$$0 = \log_3(x+9) + 2$$

$$-2 = \log_3(x+9) \text{ (log form)}$$

↑ exp      ↑ Base      ↑ ans

$$3^{-2} = x+9 \text{ (exp form)}$$

$$\left(\frac{1}{3}\right)^2 = x+9$$

$$\frac{1}{9} = x+9$$

$$\frac{1}{9} - \frac{9}{9} = x$$

$$\frac{1}{9} - \frac{81}{9} = x$$

$$-\frac{80}{9} = x$$

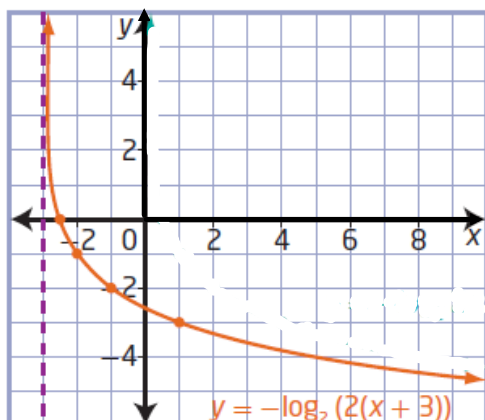
$$(-8.\bar{8}, 0)$$



## Example 2

### Reflections, Stretches, and Translations of a Logarithmic Function

- a) Use transformations to sketch the graph of the function  
 $y = -\log_2(2x + 6)$ .  $\rightarrow y = -\log_2(2(x+3))$
- b) Identify the following characteristics of the graph of the function.
- the equation of the asymptote
  - the domain and range
  - the y-intercept, if it exists
  - the x-intercept, if it exists



# Assignment

x-int ( $y=0$ )

$$y = 2 \log_3(-x+1) + 2$$

$$0 = 2 \log_3(-x+1) + 2$$

$$\frac{-2}{2} = \frac{2 \log_3(-x+1)}{2}$$

$$-1 = \log_3(-x+1) \quad (\log)$$

$\uparrow$              $\uparrow$              $\uparrow$   
 exp.        Base        ans.

$$3^{-1} = -x+1 \quad (\text{exp})$$

$$\frac{1}{3} = -x+1$$

$$\frac{1}{3} - 1 = -x$$

$$\frac{1}{3} - \frac{3}{3} = -x$$

$$\frac{-2}{-1} = \frac{-x}{-1}$$

$$\boxed{\frac{2}{3} = x}$$

$(\frac{2}{3}, 0)$

y-int ( $x=0$ )

$$y = 2 \log_3(-x+1) + 2$$

$$y = 2 \log_3(-(0)+1) + 2$$

$$y = 2 \log_3(1) + 2$$

$$y = 2(0) + 2$$

$\frac{\log 1}{\log 3} = 0$

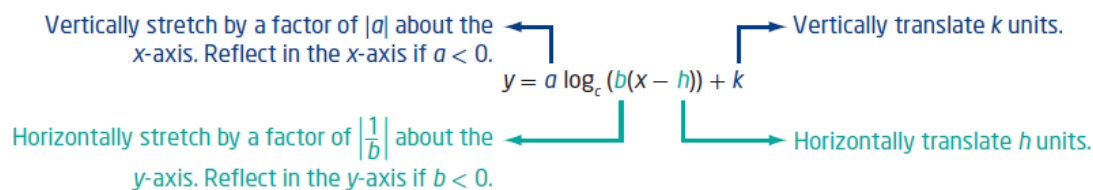
$$y = 0 + 2$$

$$\boxed{y = 2}$$

$(0, 2)$

### Key Ideas

- To represent real-life situations, you may need to transform the basic logarithmic function  $y = \log_b x$  by applying reflections, stretches, and translations. These transformations should be performed in the same manner as those applied to any other function.
- The effects of the parameters  $a$ ,  $b$ ,  $h$ , and  $k$  in  $y = a \log_c (b(x - h)) + k$  on the graph of the logarithmic function  $y = \log_c x$  are shown below.



- Only parameter  $h$  changes the vertical asymptote and the domain. None of the parameters change the range.

## Homework

Questions #1, 2, 4, 5, 8, 11 on  
page 389 - 391