Making a Complete Sketch

① Plot all points: x-int, y-int, max, min, I.P.,

② Plot all asymptotes (Check behaviour near VA.)

③ Use intervals of inc/dec and concavity to connect everything

Example:

Examine the function $f(x) = 3x^5 - 5x^3$ with respect to…

- Intercepts $f(x)$
- Symmetry
- Asymptotes (No asymptotes for polynomial functions)
- Intervals of Increase or Decrease $f'(x)$
- Local Maximum and Minimum values $f(x)$
- $f''(x)$ ~~Concavity~~ and ~~Points of Inflection~~ $f(x)$
- Sketch the Curve

$f(x) = 3x^5 - 5x^3$

$f(x) = x^3(3x^2 - 5)$

$f'(x) = 15x^4 - 15x^2$

$f'(x) = 15x^2(x^2 - 1)$

$f'(x) = 15x^2(x-1)(x+1)$

$f''(x) = 60x^3 - 30x$

$f''(x) = 30x(2x^2 - 1)$

① x-int $(y=0)$

$f(x) = x^3(3x^2 - 5)$

$0 = x^3(3x^2 - 5)$

$x^3 = 0$ | $3x^2 - 5 = 0$

$x = 0$ | $\frac{3x^2}{3} = \frac{5}{3}$

$(0,0)$ | $x^2 = \frac{5}{3}$

$x = \pm\sqrt{5/3}$

$(1.29, 0)$

$+ (-1.29, 0)$

② y-int $(x=0)$

$f(x) = 3x^5 - 5x^3$

$f(0) = 3(0)^5 - 5(0)^3$

$f(0) = 0$

$(0,0)$

Symmetry:

$f(-x) = 3(-x)^5 - 5(-x)^3$

$f(-x) = -3x^5 + 5x^3$

$f(-x) = -f(x)$

odd

③ Intervals of Inc/Dec.

$f'(x) = 15x^2(x-1)(x+1)$

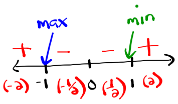
$0 = 15x^2(x-1)(x+1)$

$15x^2 = 0$ | $x - 1 = 0$ | $x + 1 = 0$

$x^2 = 0$ | $x = 1$ | $x = -1$

$x = 0$

CV: $x = -1, 0, 1$



Increasing on $(-\infty, -1) + (1, \infty)$

Decreasing on $(-1, 0) + (0, 1)$

or $(-1, 1)$

④ max @ $x = -1$

$f(x) = 3x^5 - 5x^3$

$f(-1) = 3(-1)^5 - 5(-1)^3$

$f(-1) = -3 + 5$

$f(-1) = 2$

$(-1, 2)$

⑤ min @ $x = 1$

$f(x) = 3x^5 - 5x^3$

$f(1) = 3(1)^5 - 5(1)^3$

$f(1) = 3 - 5$

$f(1) = -2$

$(1, -2)$

⑥ Intervals of Concavity:

$f''(x) = 30x(2x^2 - 1)$

$0 = 30x(2x^2 - 1)$

$30x = 0$ | $2x^2 - 1 = 0$

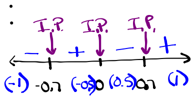
$x = 0$ | $2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{1/2}$

$x \approx \pm 0.707$

CV: $x = -0.7, 0, 0.7$



CD on $(-\infty, -0.7) + (0, 0.7)$

CU on $(-0.7, 0) + (0.7, \infty)$

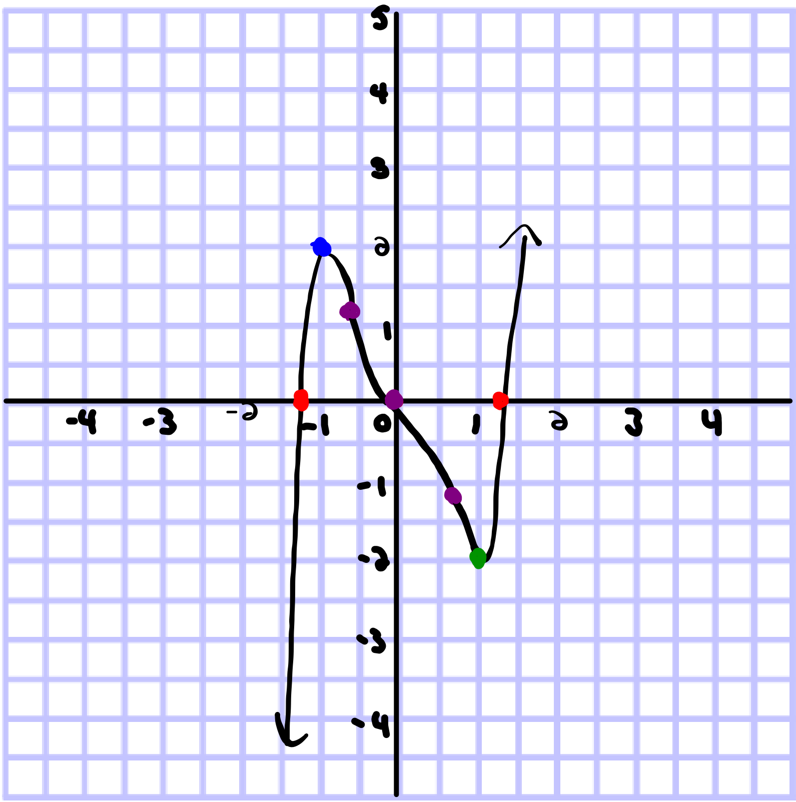
⑦ Inflection Points

$f(x) = 3x^5 - 5x^3$

$f(0) = 3(0.7)^5 - 5(0.7)^3 = -0.504 + 1.715 = 1.2$ $(-0.7, 1.2)$

$f(0) = 3(0)^5 - 5(0)^3 = 0 - 0 = 0$ $(0,0)$

$f(0.7) = 3(0.7)^5 - 5(0.7)^3 = 0.504 - 1.715 = -1.2$ $(0.7, -1.2)$


5
4
3
2
1
-4
-3
-2
-1
0
1
2
3
4
-1
-2
-3
-4

Assignment:

$f(x) = x^2 + x^3$ $\quad f'(x) = 2x + 3x^2$ $\quad f''(x) = 2 + 6x$

$f(x) = x^2(1+x)$ $\quad f'(x) = x(2+3x)$ $\quad f''(x) = 2(1+3x)$

① x-int $(y=0)$

$f(x) = x^2(1+x)$

$0 = x^2(1+x)$

$x^2 = 0$ | $1+x=0$

$x=0$ | $x=-1$

$(0,0)$ | $(-1,0)$

② y-int $(x=0)$

$f(x) = x^2 + x^3$

$f(0) = (0)^2 + (0)^3$

$f(0) = 0 + 0$

$f(0) = 0$

$(0,0)$

③ Intervals of Inc/Dec.

$f'(x) = x(2+3x)$

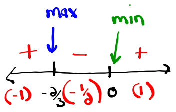
$0 = x(2+3x)$

$x = 0$ | $2+3x=0$

$3x = -2$

$x = -2/3$

CV: $x = -\frac{2}{3}, 0$



Increasing on $(-\infty, -2/3)$ + $(0, \infty)$

Decreasing on $(-2/3, 0)$

④ max @ $x = -\frac{2}{3}$

$f(x) = x^2 + x^3$

$f(\frac{-2}{3}) = (\frac{-2}{3})^2 + (\frac{-2}{3})^3$

$f(\frac{-2}{3}) = \frac{4}{9} - \frac{8}{27}$

$f(\frac{-2}{3}) = \frac{12}{27} - \frac{8}{27} = \frac{4}{27}$

$(-\frac{2}{3}, \frac{4}{27})$ or $(-0.\bar{6}, 0.15)$

⑤ min @ $x = 0$

$f(x) = x^2 + x^3$

$f(0) = (0)^2 + (0)^3$

$f(0) = 0 + 0 = 0$

$(0,0)$

⑥ Intervals of Concavity:

$f''(x) = 2(1+3x)$

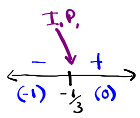
$0 = 2(1+3x)$

$2 \neq 0$ | $1+3x=0$

$3x = -1$

$x = -\frac{1}{3}$

CV: $x = -\frac{1}{3}$



CD on $(-\infty, -\frac{1}{3})$

CU on $(-\frac{1}{3}, \infty)$

⑦ Inflection Point: @ $x = -\frac{1}{3}$

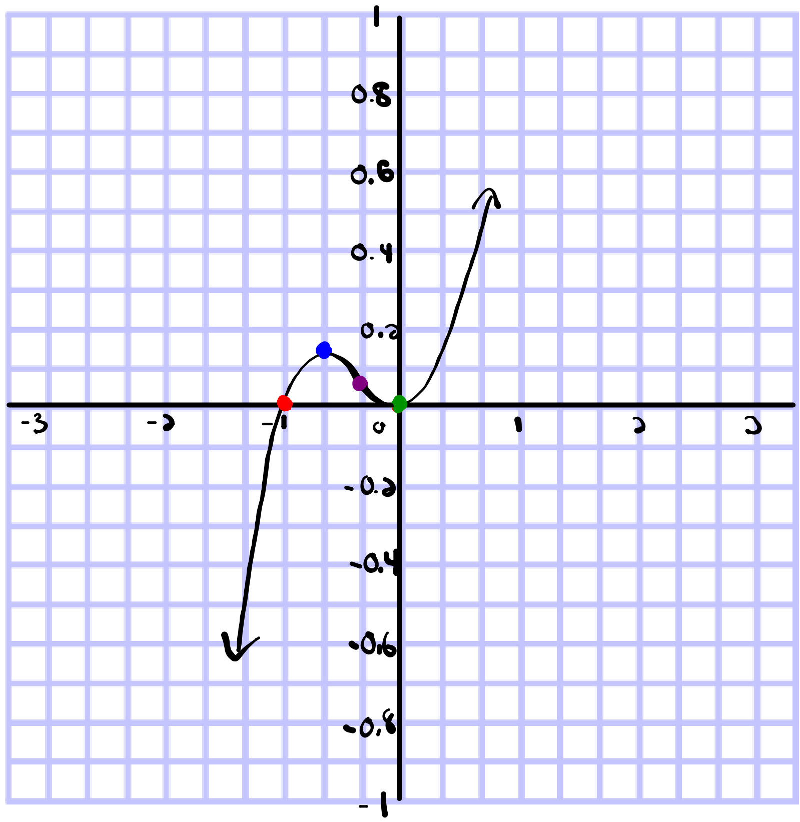
$f(x) = x^2 + x^3$

$f(\frac{-1}{3}) = (\frac{-1}{3})^2 + (\frac{-1}{3})^3$

$f(-\frac{1}{3}) = \frac{1}{9} - \frac{1}{27}$

$f(-\frac{1}{3}) = \frac{3}{27} - \frac{1}{27} = \frac{2}{27}$

$(-\frac{1}{3}, \frac{2}{27})$ or $(-0.\bar{3}, 0.07)$


1
0.8
0.6
0.4
0.2
-3
-2
-1
0
1
2
3
-0.2
-0.4
-0.6
-0.8
-1

homework

Examine the function $f(x)=\frac{x^2}{1-x^2}$ with respect to… $f'(x)=\frac{2x}{(1-x^2)^2}$

$f''(x)=\frac{2(1+3x^2)}{(1-x^2)^3}$

- Intercepts $f(x)$
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

① x-int $(y=0)$

$f(x)=\frac{x^2}{1-x^2}$

$0=\frac{x^2}{1-x^2}$

$0=x^2$

$0=x$

$(0,0)$

② y-int $(x=0)$

$f(x)=\frac{x^2}{1-x^2}$

$f(0)=\frac{(0)^2}{1-(0)^2}$

$f(0)=\frac{0}{1}=0$

$(0,0)$

Symmetry:

$f(-x)=\frac{(-x)^2}{1-(-x)^2}$

$f(-x)=\frac{x^2}{1-x^2}$

$f(-x)=f(x)$

Even

③ Vertical Asymptote:
(zeroes of the denominator)

$f(x)=\frac{x^2}{1-x^2}$

VA: $1-x^2=0$

$(1-x)(1+x)=0$

$1-x=0$ | $1+x=0$

$\boxed{1=x}$ | $\boxed{x=-1}$

④ Horizontal Asymptote:

$\lim_{x\to\infty}\frac{x^2}{1-x^2}=\frac{1}{-1}=-1$

$\boxed{y=-1}$

⑤ Intervals of Inc/Dec.

$f'(x)=\frac{2x}{(1-x^2)^2}$

$2x=0$

$x=0$

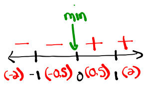
$(1-x^2)^2=0$

$1-x^2=0$

$1=x^2$

$\pm1=x$

CV: $x=-1,0,1$



Increasing on $(0,\infty)$

Decreasing on $(-\infty,0)$

⑥ min @ $x=0$

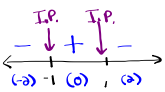
$f(x)=\frac{x^2}{1-x^2}$

$f(0)=\frac{(0)^2}{1-(0)^2}=0$

$(0,0)$

⑦ Intervals of concavity

$f''(x)=\frac{2(1+3x^2)}{(1-x^2)^3}$



$2(1+3x^2)=0$

$1+3x^2=0$

$3x^2=-1$

$x^2=-\frac{1}{3}$

Not Possible

(Numerator is always positive)

$(1-x^2)^3=0$

$1-x^2=0$

$1=x^2$

$\pm1=x$

CD on $(-\infty,-1)+(1,\infty)$

CU on $(-1,1)$

⑧ Inflection Points:

when $x=-1$

$f(x)=\frac{x^2}{1-x^2}$

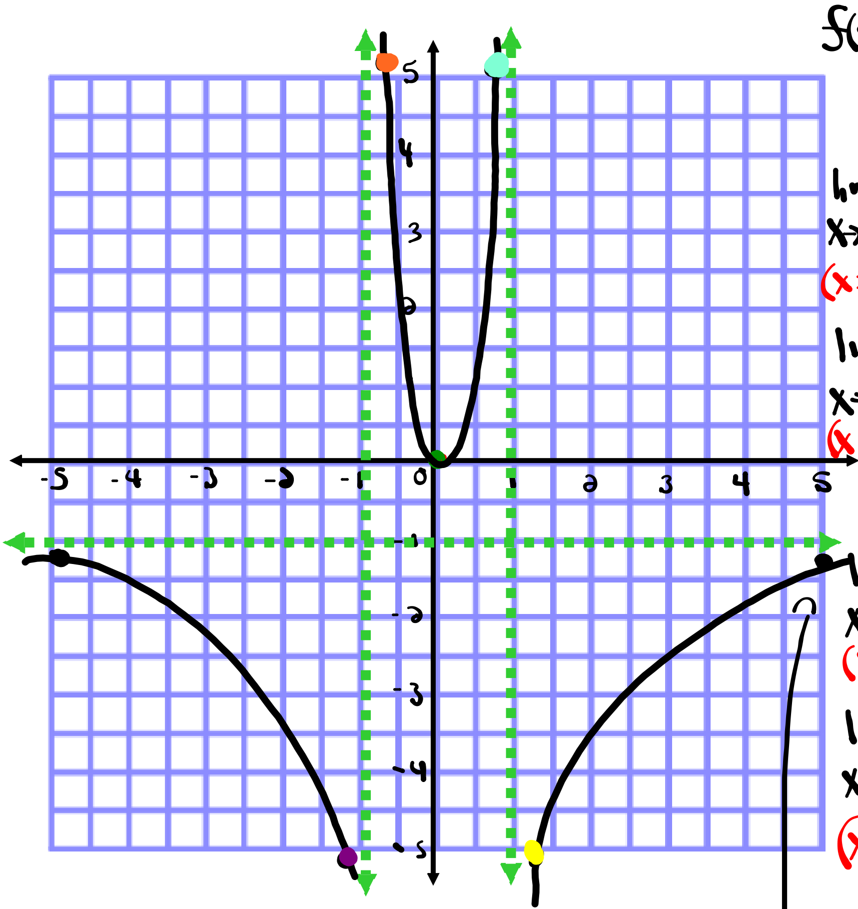
$f(-1)=\frac{(-1)^2}{1-(-1)^2}=\frac{1}{0}=$ und.

when $x=1$

$f(x)=\frac{x^2}{1-x^2}$

$f(1)=\frac{(1)^2}{1-(1)^2}=\frac{1}{0}=$ und

* No Inflection Points $x=\pm1$ are the Vertical Asymptotes.



$f(x) = \dfrac{x^2}{(1-x)(1+x)}$

$\lim_{x \to -1^-} f(x) = \dfrac{(+)}{(+)(-)} = -\infty$

$(x = -1.1)$

$\lim_{x \to -1^+} f(x) = \dfrac{(+)}{(+)(+)} = +\infty$

$(x = -0.9)$

$\lim_{x \to 1^-} f(x) = \dfrac{(+)}{(+)(+)} = +\infty$

$(x = 0.9)$

$\lim_{x \to 1^+} f(x) = \dfrac{(+)}{(-)(+)} = -\infty$

$(x = 1.1)$

?

$f(5) = \dfrac{(5)^2}{1-(5)^2} = \dfrac{25}{-24}$

$$f(x) = \frac{x^2}{1-x^2}$$

$$f'(x) = \frac{2x(1-x^2) + 2x(x^2)}{(1-x^2)^2}$$

$$f'(x) = \frac{2x - 2x^3 + 2x^3}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

$$f''(x) = \frac{2(1-x^2)^2 - 2x(2)(1-x^2)(-2x)}{(1-x^2)^4}$$

$$f''(x) = \frac{2(1-x^2)^2 + 8x^2(1-x^2)}{(1-x^2)^4}$$

$$f''(x) = \frac{2(1-x^2)\left[(1-x^2) + 4x^2\right]}{(1-x^2)^4} =$$

$$f''(x) = \frac{2\cancel{(1-x^2)}(1+3x^2)}{(1-x^2)^{\cancel{4}\,3}} = \frac{2(1+3x^2)}{(1-x^2)^3}$$

Review:

① $f(x) = \frac{8(x-2)}{x^2}$ $f'(x) = \frac{-8(x-4)}{x^3}$ $f''(x) = \frac{16(x-6)}{x^4}$

Intercepts:

① x-int (y=0)

$\frac{0}{1} = \frac{8(x-2)}{x^2}$

$8(x-2) = 0$

$x-2=0$

$x=2$

$(2,0)$

② y-int (x=0)

$f(0) = \frac{8(0-2)}{(0)^2}$

$f(0) = \frac{-16}{0}$

$f(0)$ = undefined

No y-int

Symmetry:

$f(-x) = \frac{8(-x-2)}{(-x)^2}$

$f(-x) = \frac{8(-x-2)}{x^2}$

$f(-x) = \frac{-8(x+2)}{x^2}$

No symmetry

③ VA: (zeros of denom)

$f(x) = \frac{8(x-2)}{x^2}$

$x^2 = 0$

$\boxed{x=0}$

Check behaviour of f(x) near VA:

$\lim_{x\to 0^-} f(x) = \frac{(-)}{(+)} = -\infty$

$(x=-0.01)$

$\lim_{x\to 0^+} f(x) = \frac{(-)}{(+)} = -\infty$

$(x=0.01)$

④ HA:

$f(x) = \frac{8(x-2)}{x^2} = \frac{8x-16}{x^2}$

$\lim_{x\to\infty} \frac{8x-16}{x^2} = 0$

$\boxed{y=0}$

⑤ Intervals of Inc/Dec

$f'(x) = \frac{-8(x-4)}{x^3}$

CV:

$-8(x-4)=0$ | $x^3=0$

$x-4=0$ | $x=0$

$x=4$

min max

− + −

(-1) 0 (1) 4 (5)

Increasing on $(0,4)$

Decreasing on $(-\infty,0)$ + $(4,\infty)$

⑥ Max/Min:

$f(x) = \frac{8(x-2)}{x^2}$

$f(0) = \frac{8(0-2)}{(0)^2}$ = undefine

no min

x=0 is the VA

$f(4) = \frac{8(4-2)}{(4)^2} = \frac{16}{16} = 1$

max at $(4,1)$

⑦ Concavity:

$f''(x) = \frac{16(x-6)}{x^4}$

CV:

$16(x-6)=0$ | $x^4=0$

$x-6=0$ | $x=0$

$x=6$

I.P.

− − +

(-1) 0 (1) 6 (7)

CU on $(6,\infty)$

CD on $(-\infty,0)$ + $(0,6)$

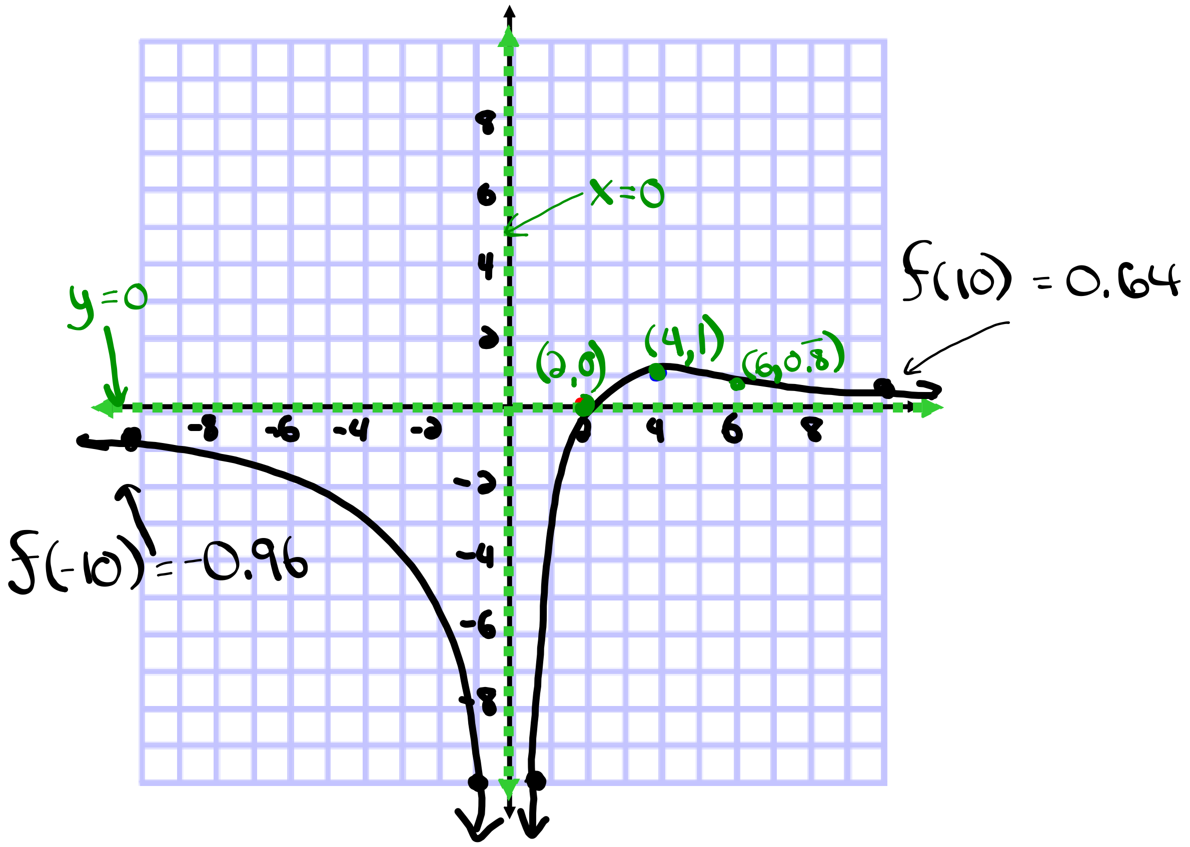
⑧ Inflection Point

$f(x) = \frac{8(x-2)}{x^2}$

$f(6) = \frac{8(6-2)}{6^2} = \frac{32}{36}$

$(6, 0.\overline{8})$

Review:



Review:

Examine the function $f(x) = x^4 - 4x^3$ with respect to…

- Intercepts
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

$f(x) = x^4 - 4x^3$ $\quad$ $f'(x) = 4x^3 - 12x^2$ $\quad$ $f''(x) = 12x^2 - 24x$

$f(x) = x^3(x-4)$ $\quad$ $f'(x) = 4x^2(x-3)$ $\quad$ $f''(x) = 12x(x-2)$

① x-int $(y=0)$

$f(x) = x^3(x-4)$

$0 = x^3(x-4)$

$x^3 = 0$ | $x - 4 = 0$

$x = 0$ | $x = 4$

$(0,0)$ | $(4,0)$

② y-int $(x=0)$

$f(x) = x^4 - 4x^3$

$f(0) = (0)^4 - 4(0)^3$

$f(0) = 0$

$(0,0)$

Symmetry:

$f(-x) = (-x)^4 - 4(-x)^3$

$f(-x) = x^4 + 4x^3$

No symmetry

③ Intervals of Inc/Dec.

$f'(x) = 4x^2(x-3)$

$4x^2 = 0$ | $x - 3 = 0$

$x^2 = 0$ | $x = 3$

$x = 0$

CV: $x = 0, 3$

min

$-$ $\quad$ $-$ $\quad$ $+$

(-1) $\quad$ 0 $\quad$ (1) $\quad$ 3 $\quad$ (4)

Decreasing on $(-\infty, 3)$

Increasing on $(3, \infty)$

④ min @ $\underline{x = 3}$

$f(x) = x^4 - 4x^3$

$f(3) = (3)^4 - 4(3)^3$

$f(3) = 81 - 108$

$f(3) = -27$

$(3, -27)$

⑤ No local max

⑥ Intervals of Concavity:

$f''(x) = 12x(x-2)$

$12x = 0$ | $x - 2 = 0$

$x = 0$ | $x = 2$

CV: $x = 0, 2$

I.P. $\quad$ I.P.

$+$ $\quad$ $-$ $\quad$ $+$

(-1) $\quad$ 0 $\quad$ (1) $\quad$ 2 $\quad$ (3)

CD on $(0, 2)$

CU on $(-\infty, 0)$ and $(2, \infty)$

⑦ Inflection points:

when $\underline{x = 0}$

$f(x) = x^4 - 4x^3$

$f(0) = (0)^4 - 4(0)^3$

$f(0) = 0$

$(0,0)$

when $\underline{x = 2}$

$f(x) = x^4 - 4x^3$

$f(2) = (2)^4 - 4(2)^3$

$f(2) = 16 - 32$

$f(2) = -16$

$(2, -16)$

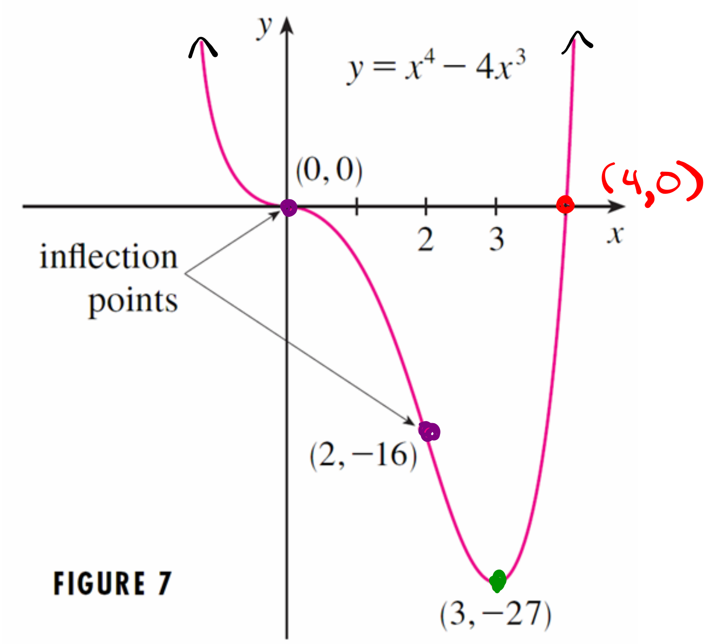


FIGURE 7

homework

Examine the function $f(x) = \frac{x^2}{x-7}$ with respect to…

- Intercepts
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

① x-int (y=0)

$f(x) = \frac{x^2}{x-7}$

$(x-7)\, 0 = \frac{x^2}{\cancel{x-7}} \cancel{(x-7)}$

$0 = x^2$

$0 = x$

(0,0)

② y-int (x=0)

$f(x) = \frac{x^2}{x-7}$

$f(0) = \frac{(0)^2}{(0)-7} = \frac{0}{-7} = 0$

$y = 0$

(0,0)

③ Symmetry:

$f(x) = \frac{x^2}{x-7}$

$f(-x) = \frac{(-x)^2}{(-x)-7}$

$f(-x) = \frac{x^2}{-x-7}$

No symmetry

④ VA: (denom = 0)

$x - 7 = 0$

$x = 7$

$\lim_{x\to 7^-} \frac{x^2}{x-7} = \frac{(+)}{(-)} = -\infty$ (6.9)

$\lim_{x\to 7^+} \frac{x^2}{x-7} = \frac{(+)}{(+)} = +\infty$ (x=7.1)

⑤ SA:

$$\begin{array}{r} x+7 \\ x-7\,\overline{)\,x^2} \\ -(x^2-7x) \\ \hline 7x \\ -(7x-49) \\ \hline 49\,R \end{array}$$

$y = x + 7$

$m = \frac{1}{1}$ $\frac{\text{rise}}{\text{run}}$

$b = 7$ y-int

⑥ Intervals of Inc/Dec:

$f'(x) = \frac{x(x-14)}{(x-7)^2}$

max neither min

+ − − +

(−1) 0 (1) 7 (8) 14 (15)

CV: $x = 0$ | $x - 14 = 0$, $x = 14$ | $(x-7)^2 = 0$, $x - 7 = 0$, $x = 7$

Increasing on $(-\infty, 0) + (14, \infty)$

$x < 0$ + $x > 14$

Decreasing on $(0, 14)$

$0 < x < 14$

⑦ Local max/min

$f(x) = \frac{x^2}{x-7}$

when $x=0$

$f(0) = \frac{(0)^2}{(0)-7} = \frac{0}{-7} = 0$

(0,0)

local max @ (0,0)

When $x = 14$

$f(14) = \frac{(14)^2}{(14)-7} = \frac{196}{7} = 28$

(14,28)

local min @ (14,28)

⑧ Intervals of Concavity:

$f''(x) = \frac{98}{(x-7)^3}$

I.P.

− +

(6) (7) (8)

CV: $98 \neq 0$ | $(x-7)^3 = 0$, $x - 7 = 0$, $x = 7$

Concave down on $(-\infty, 7)$

$x < 7$

Concave up on $(7, \infty)$

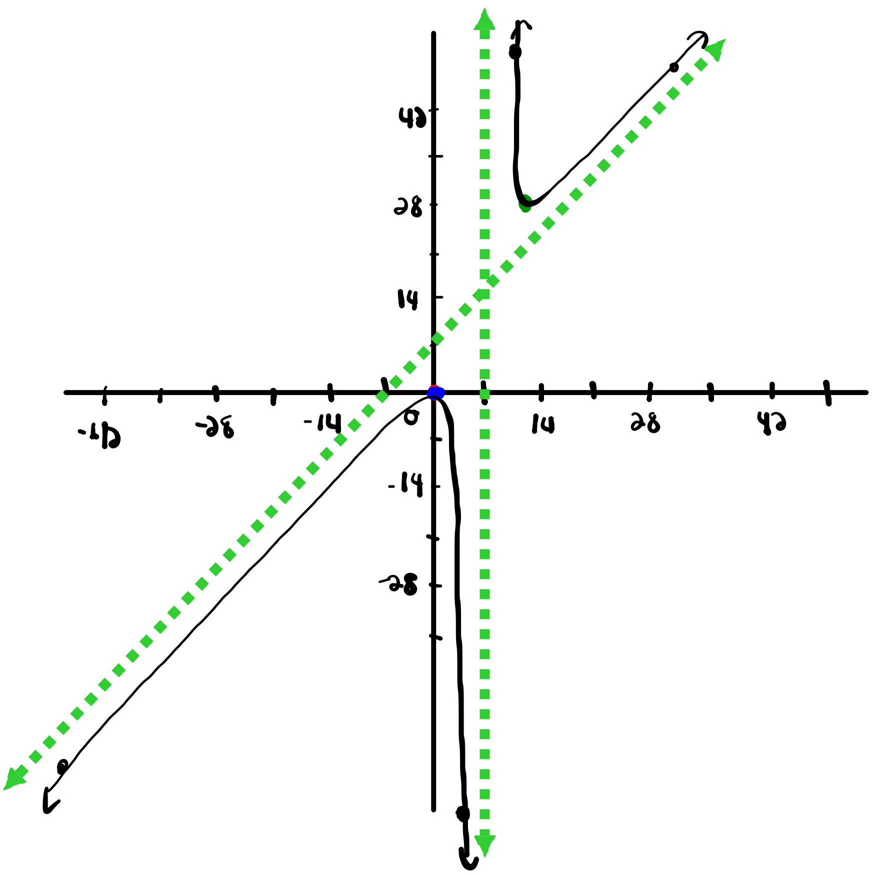
$x > 7$

⑨ I.P. (x=7)

$f(x) = \frac{x^2}{x-7}$

$f(7) = \frac{(7)^2}{(7)-7} = \frac{49}{0} = \text{DNE}$

x=7 is the vertical asymptote.



42
28
14
-14
-28
-42
-28
-14
0
14
28
42

Review:

$f(x) = x^4 - 3x^3 + 3x^2 - x$

① x-int $(y=0)$

$0 = x^4 - 3x^3 + 3x^2 - x$

$0 = x(x^3 - 3x^2 + 3x - 1)$ syn. sub.

$(1)^3 - 3(1)^2 + 3(1) - 1$

$1 - 3 + 3 - 1$

$$\begin{array}{c|cccc} 1 & 1 & -3 & 3 & -1 \\ & & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & \end{array}$$

$0 = x(x-1)(x^2 - 2x + 1)$ simple tri.

$-1 \times -1 = 1$

$-1 + -1 = -2$

$0 = x(x-1)(x-1)(x-1)$

$x = 0$	$x - 1 = 0$
	$x = 1$
$(0,0)$	$(1,0)$

② y-int $(x=0)$

$f(x) = x^4 - 3x^3 + 3x^2 - x$

$f(0) = (0)^4 - 3(0)^3 + 3(0)^2 - (0) = 0$

$(0,0)$

Intervals of Inc/Dec:

$f(x) = x^4 - 3x^3 + 3x^2 - x$

$f'(x) = 4x^3 - 9x^2 + 6x - 1$ syn. sub

$4(1)^3 - 9(1)^2 + 6(1) - 1$

$4 - 9 + 6 - 1$

$$\begin{array}{c|cccc} 1 & 4 & -9 & 6 & -1 \\ & & 4 & -5 & 1 \\ \hline & 4 & -5 & 1 & \end{array}$$

$f'(x) = (x-1)(4x^2 - 5x + 1)$ Hard Trinomial

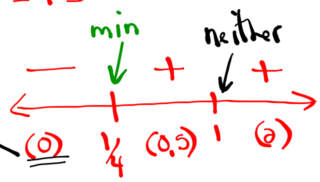
$f'(x) = (x-1)(x - \frac{1}{4})(x - \frac{4}{4})$

$-1 \times -4 = 4$

$-1 + -4 = -5$

$f'(x) = (x-1)(4x-1)(x-1)$

$f'(x) = (x-1)^2(4x-1)$



CV:

$(x-1)^2 = 0$	$4x - 1 = 0$
$x - 1 = 0$	$4x = 1$
$x = 1$	$x = \frac{1}{4}$

$x = \frac{1}{4}, 1$

Decreasing on $(-\infty, \frac{1}{4})$

$x < \frac{1}{4}$

Increasing on $(\frac{1}{4}, \infty)$

$x > \frac{1}{4}$