1.3 Using Reasoning to Find a Counterexample to a Conjecture

GOAL

Invalidate a conjecture by finding a contradiction.

EXPLORE...

- Six, twelve, ten, one, fifty ...
  Conjecture: All but one of the vowels (a, e, i, o, u, and y) are used to spell numbers. Gather evidence to support or deny this conjecture.

SAMPLE ANSWER

The number words to 100 contain all the vowels except *a*.

zero	ten	twenty
one	eleven	thirty
two	twelve	forty
three	thirteen	fifty
four	fourteen	sixty
five	fifteen	seventy
six	sixteen	eighty
seven	seventeen	ninety
eight	eighteen	hundred
nine	nineteen	thousand

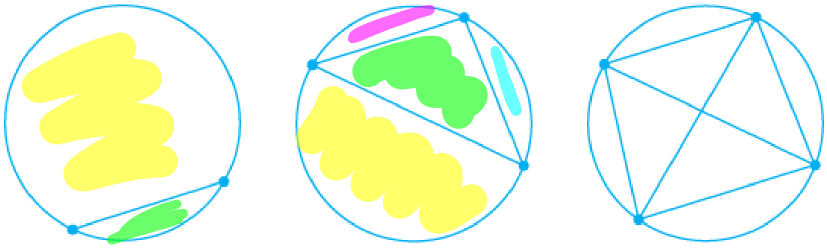
(counter example)

These number words are used for all the numbers to 999. The word thousand is the first number word that contains the vowel *a*.

Revised Conjecture: All of the number words from 1 to 999 contain all vowels except "a"

LEARN ABOUT the Math

Kerry created a series of circles. Each circle had points marked on its circumference and joined by chords.



As the number of points on the circumference increased, Kerry noticed a pattern for the number of regions created by the chords.

Number of Points	2	3	4	5	6	7	8
Number of Regions	2	4	8	16	32	64	128

. . . . ?

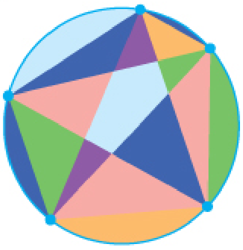
She made the following conjecture: As the number of connected points on the circumference of a circle increases by 1, the number of regions created within the circle increases by a factor of 2.

How can Kerry test the validity of her conjecture?

EXAMPLE 1 Testing a conjecture

Gather more evidence to test Kerry's conjecture.

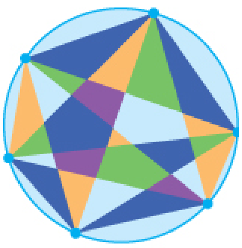
Zohal's Solution



Number of Points	2	3	4	5
Number of Regions	2	4	8	16

I drew another circle and identified five points on its circumference. Then I joined the pairs of points with chords. I coloured the resulting regions to make them easier to count.

My diagram had 16 regions. This supported Kerry's conjecture because the pattern for the resulting regions was 2^1, 2^2, 2^3, 2^4.



Number of Points	2	3	4	5	6
Number of Regions	2	4	8	16	31

counterexample

I drew another circle and identified six points on its circumference. Then I joined the pairs of points with chords and coloured the regions.

When I counted, I got only 31 regions, not 2^5 or 32 as Kerry's conjecture predicts.

The number of regions did not increase by a factor of 2. This **counterexample** disproves Kerry's conjecture.

counterexample
An example that invalidates a conjecture.

Reflecting

A. Why do you think Zohal started her development of further evidence by using five points on the circumference of a circle?

B. Why is only one counterexample enough to disprove a conjecture?

Answers

A. I think Zohal started her samples with five points on the circle to continue the pattern in Kerry's evidence. If there are regular increments in the pattern, then possible counterexamples in the lesser values might be found. This would avoid the need to work with greater numbers of points and the challenge of counting the resulting regions.

B. One counterexample is enough to disprove a conjecture because the counterexample shows a case when the conjecture is not valid. Once a counterexample is found, the conjecture is no longer valid.

APPLY the Math

EXAMPLE 2 Connecting to previous conjectures

In Lesson 1.1, page 9, Francesca and Steffan made conjectures about the difference between consecutive squares. Ex: $36 - 25 = 11$

Steffan's conjecture: The difference between consecutive perfect squares is always an odd number.

Francesca's conjecture: The difference between consecutive perfect squares is always a prime number.

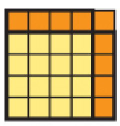
How can these conjectures be tested?

Luke's Solution: Communicating about Steffan's conjecture and more trials

Steffan's conjecture was true for the pairs of consecutive squares he chose: 2 × 2 and 3 × 3, 3 × 3 and 4 × 4, and 5 × 5 and 6 × 6.

$4 - 1 = 3$

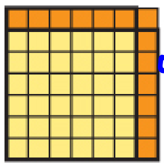
First, I tried 1 × 1 and 2 × 2. I made the same tile squares as Steffan. When I took away the yellow square, I was left with a pair of tiles that shared an edge with the yellow square and a single tile in the top right corner.



$25 - 16 = 9$

Next, I chose 4 × 4 and 5 × 5, since Steffan had skipped over these values. I was left with two groups of tiles, each with the same value as a side of the yellow square, plus one extra tile in the top right corner.

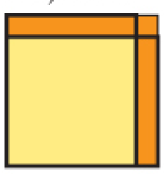
Counterexample to Fran . . . .



$49 - 36 = 13$

I tried consecutive squares of 6 × 6 and 7 × 7. The difference again showed the same pattern: two groups of tiles, each with the same value as a side of the yellow square, plus a single tile in the top right corner.

These three examples support Steffan's conjecture.



I visualized what the difference would look like for any pair of consecutive squares. There would always be two groups of orange tiles, each with the same value as a side of the smaller yellow square, plus one unpaired orange square in the corner. The total value of the two equal groups would always be an even number, since 2 times any number is even. The unpaired tile would make the difference odd.

All this evidence strengthens the validity of Steffan's conjecture. However, it doesn't prove the conjecture since I haven't tried all the possible cases.

EXAMPLE 2 Connecting to previous conjectures

In Lesson 1.1, page 9, Francesca and Steffan made conjectures about the difference between consecutive squares.

Steffan's conjecture: The difference between consecutive perfect squares is always an odd number.

Francesca's conjecture: The difference between consecutive perfect squares is always a prime number.

How can these conjectures be tested?

Pierre's Solution: Connecting more evidence to Francesca's conjecture

Francesca used the consecutive squares of 1 and 2, 3 and 4, and 8 and 9.

I chose values so I could start to fill the gaps between the values that Francesca chose.

$3^2 - 2^2 = 5$

Five is a prime number.

$5^2 - 4^2 = 9$

The next gap was 4 and 5. Nine is not a prime number.

counterexample

Francesca's conjecture, that the difference between consecutive squares is always a prime number, was disproved since a counterexample was found.

EXAMPLE 2 Connecting to previous conjectures

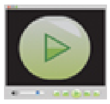
In Lesson 1.1, page 9, Francesca and Steffan made conjectures about the difference between consecutive squares.

Steffan's conjecture: The difference between consecutive perfect squares is always an odd number.

Francesca's conjecture: The difference between consecutive perfect squares is always a prime number.

How can these conjectures be tested?

Your Turn



a) Find another counterexample to Francesca's conjecture.

b) Can you find a counterexample to Steffan's conjecture? Explain.

Answers

a) $8^2 - 7^2 = 15$

15 is not a prime number.

b) I can't find a counterexample to Steffan's conjecture because Luke's visualization presents a strong argument for the conjecture being valid in all cases. Even though Luke's visualization does not prove the conjecture for all cases, it strengthens my belief that the pattern will be repeated in all cases.

EXAMPLE 3 Using reasoning to find a counterexample to a conjecture

Matt found an interesting numeric pattern:

$1 \cdot 8 + 1 = \mathbf{9}$

$12 \cdot 8 + 2 = \mathbf{98}$

$123 \cdot 8 + 3 = \mathbf{987}$

$1234 \cdot 8 + 4 = \mathbf{9876}$

Matt thinks that this pattern will continue.

Search for a counterexample to Matt's conjecture.

Kublu's Solution

$\mathbf{1} \cdot 8 + \mathbf{1} = 9$

$\mathbf{12} \cdot 8 + \mathbf{2} = 98$

$\mathbf{123} \cdot 8 + \mathbf{3} = 987$

$\mathbf{1234} \cdot 8 + \mathbf{4} = 9876$

The pattern seemed to be related to the first factor (the factor that wasn't 8), the number that was added, and the product.

	A	B
1	1 · 8 + 1	9
2	12 · 8 + 2	98
3	123 · 8 + 3	987
4	1234 · 8 + 4	9876
5	12345 · 8 + 5	98765
6	123456 · 8 + 6	987654
7	1234567 · 8 + 7	9876543
8	12345678 · 8 + 8	98765432
9	123456789 · 8 + 9	987654321

I used a spreadsheet to see if the pattern continued.

The spreadsheet showed that it did.

$12345678\mathbf{910} \cdot 8 + \mathbf{10} = 98\ 765\ 431\ 290$ counterexample

$123456789\mathbf{0} \cdot 8 + \mathbf{10} = 9\ 876\ 543\ 130$

$12345678\mathbf{910} \cdot 8 + \mathbf{0} = 98\ 765\ 431\ 280$

$123456789\mathbf{0} \cdot 8 + \mathbf{0} = 9\ 876\ 543\ 120$

When I came to the tenth step in the sequence, I had to decide whether to use 10 or 0 in the first factor and as the number to add. I decided to check each way that 10 and 0 could be represented.

The pattern holds true until 9 of the 10 digits are included. At the tenth step in the sequence, a counterexample is found.

Since the pattern did not continue, Matt's conjecture is invalid.

Revised conjecture: When the value of the addend is 1 to 9, the pattern will continue.

I decided to revise Matt's conjecture by limiting it.

EXAMPLE 3 Using reasoning to find a counterexample to a conjecture

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Matt thinks that this pattern will continue.
Search for a counterexample to Matt's conjecture.

Your Turn

If Kublu had not found a counterexample at the tenth step, should she have continued looking? When would it be reasonable to stop gathering evidence if all the examples supported the conjecture? Justify your decision.

Answer

If Kublu had not found a counterexample at the 10th step, she could have still stopped there. With the quantity of evidence found to support the conjecture, and a two-digit number further validating the conjecture, the conjecture could be considered strongly supported. If she had wanted to do one more example, then it might have been logical to try a three-digit number to see if the conjecture was valid in that case.

In Summary

Key Ideas

- Once you have found a counterexample to a conjecture, you have disproved the conjecture. This means that the conjecture is invalid.
- You may be able to use a counterexample to help you revise a conjecture.

Need to Know

- A single counterexample is enough to disprove a conjecture.
- Even if you cannot find a counterexample, you cannot be certain that there is not one. Any supporting evidence you develop while searching for a counterexample, however, does increase the likelihood that the conjecture is true.

Assignment: pages 22-23

Questions: 1abdfg, 3, 4, 5, 7,10,12,14

SOLUTIONS => 1.3 Using Reasoning to Find a Counterexample to a Conjecture

1. Show that each statement is false by finding a counterexample.

a) A number that is not negative is positive.

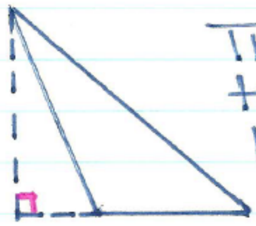
Counterexample: 0 is a number that is not negative or positive.

b) All prime numbers are odd.

Counterexample: 2 is a prime number that is not odd.

d) The height of a triangle lies inside the triangle.

Counterexample:



The height of a triangle can lie outside of the triangle.

f) The square root of a number is always less than the number.

Counterexample: $\sqrt{0.04} = 0.2$

↳ larger than 0.04

g) The sum of two numbers is always greater than the greater of the two numbers.

Counterexample: $-20 + 15 = -5$

↳ not greater than 15.

3. Jim claims that whenever you multiply two whole numbers, the product is greater than either of the two factors. Do you agree or disagree? Justify your decision.

SOLUTION

I disagree with this statement since a counterexample can be found.

Counterexample: $1 \times 25 = 25$

↳ not greater than 25

4. Rachelle claims that the sum of a multiple of 3 and a multiple of 6 must be a multiple of 6. Do you agree or disagree? Justify your decision.

SOLUTION

I disagree with this statement since a counterexample can be found.

Counterexample: $\underbrace{27}_{\text{Multiple of 3}} + \underbrace{18}_{\text{Multiple of 6}} = \underbrace{45}_{\text{Not a multiple of 6.}}$

5. Hannah examined these multiples of 9: 18, 45, 63, 27, 81, 108, 216. She claimed that the sum of the digits in any multiple of 9 will add to 9. Do you agree or disagree? Justify your decision.

SOLUTION

I disagree with this statement since a counterexample can be found.

Counterexample: $99 \rightarrow$ Multiple of 9 (9x11)
$9+9=18$ (Sum of digits)
$18 \neq 9$

7. Claire noticed that the digits 4, 5, 6, and 7 could be used to express each value from 1 to 5 as shown to the right. She conjectured that these digits could be used to express each value from 1 to 20. Explain, with examples, whether Claire's conjecture is reasonable.

Number	Expression
1	$\frac{7-5}{6-4}$
2	$7-6+5-4$
3	$\frac{6(7-5)}{4}$
4	$7+6-5-4$
5	$5(\sqrt{64}-7)$

SOLUTION

Claire's conjecture seems reasonable because so many combinations are possible. I listed a few examples on the right.

Number	Expression
6	$\frac{6}{\sqrt{4}}(7-5)$
7	$(4\times5)-(6+7)$
10	$\frac{6(5)}{7-4}$
19	$4(5)-7+6$

10. Patrice studied the following table and made this conjecture: The sums of the squares of integers separated by a value of 2 will always be even.

$(-1)^2 + 1^2 = 2$	$2^2 + 4^2 = 20$	$(-3)^2 + (-5)^2 = 34$	$4^2 + 6^2 = 52$	$0^2 + 2^2 = 4$

Is this conjecture reasonable? Explain.

SOLUTION

Patrice's conjecture is reasonable. Integers separated by a value of 2 will either both be even or both be odd, and therefore their squares will both be even or both be odd. If you add two even numbers together, the result will be even. (ie. $64 + 100 = 164$) If you add two odd numbers together, the result will be even. (ie. $63 + 81 = 144$)

12. Amy made the following conjecture: When any number is multiplied by itself, the product will be greater than the starting number. For example, in $2 \cdot 2 = 4$, the product 4 is greater than the starting number 2. Meagan disagreed with Amy's conjecture, however, because $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ and $\frac{1}{4}$ is less than $\frac{1}{2}$.

How could Amy's conjecture be improved? Explain the change(s) you would make.

SOLUTION Amy's conjecture could be changed to "When any number greater than 1 is multiplied by itself, it will be greater than the starting number."

14. Tim conjectured that all natural numbers can be written as the sum of consecutive natural numbers, based on these examples:

$10 = 1 + 2 + 3 + 4$ $12 = 3 + 4 + 5$
$9 = 4 + 5$ $94 = 22 + 23 + 24 + 25$

Do you agree or disagree with Tim's conjecture? Justify your decision.

SOLUTION

I disagree with Tim's conjecture since a counterexample can be found.

Counterexample: The number 2 cannot be written as the sum of consecutive numbers.

Attachments

PM11-1s3.gsp

1s3e2 finalt.mp4

1s3e3 final.mp4