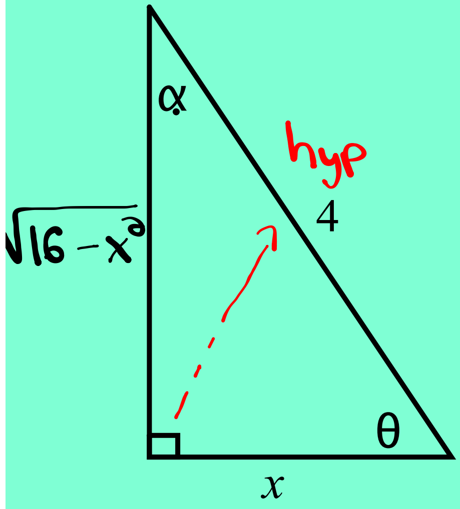
Warm Up

Using the diagram shown determine an expression for each of the following:



$\sin\theta = \frac{o}{h} = \frac{\sqrt{16-x^2}}{4}$ $\sec\alpha = \frac{h}{a} = \frac{4}{\sqrt{16-x^2}}$

$\theta = \sin^{-1}\left(\frac{\sqrt{16-x^2}}{4}\right)$

$\tan\alpha = \frac{o}{a} = \frac{x}{\sqrt{16-x^2}}$ $\tan\theta = \frac{o}{a} = \frac{\sqrt{16-x^2}}{x}$

$\cos^{-1}\overset{\frac{a}{h}}{\left(\frac{x}{4}\right)} = \theta$ $\sec^{-1}\left(\frac{4}{\sqrt{16-x^2}}\right)^{\frac{h}{a}} = \alpha$

$\cos\theta = \frac{x}{4}$ $\sec\alpha = \frac{4}{\sqrt{16-x^2}}$

$a^2+b^2=c^2$

$x^2+b^2=4^2$

$x^2+b^2=16$

$b^2=16-x^2$

$b=\pm\sqrt{16-x^2}$

$\boxed{b=\sqrt{16-x^2}}$

Derivatives of Transcendental Functions

transcendental functions

(*mathematics*) Functions which cannot be given by any algebraic expression involving only their variables and constants.

Examples include the functions log *x*, sin *x*, cos *x*, e^x and any functions containing them.

Inverse Trigonometric Functions

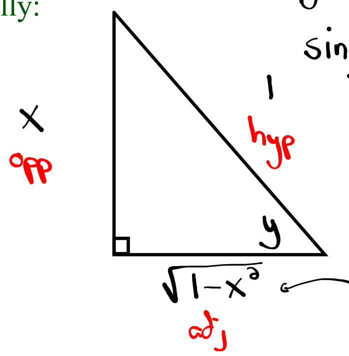
Let's review the definition of an inverse trigonometric function:

$$y = \sin^{-1} x \qquad \text{or} \qquad y = Arc\sin x$$

What do the above statements mean verbally?

"y represents the angle whose sine ratio is equal to x."

Express this visually:



$y = \sin^{-1}(x)$

$\sin y = \frac{x}{1}$ $\frac{opp}{hyp}$

$x^2 + b^2 = 1^2$

$b^2 = 1 - x^2$

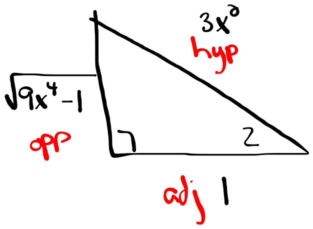
$b = \sqrt{1-x^2}$

Example:

- Represent the inverse trigonometric function $z = \sec^{-1}(3x^2)$ with a diagram.
- Evaluate: $y = \tan[\sec^{-1}(3x^2)]$

① $\underline{z} = \sec^{-1}(3x^2)$

$\sec z = \frac{3x^2}{1}$ $\frac{hyp}{adj}$



$a^2 + b^2 = c^2$

$a^2 + (1)^2 = (3x^2)^2$

$a^2 + 1 = 9x^4$

$a^2 = 9x^4 - 1$

$a = \sqrt{9x^4-1}$

② $y = \tan[\underline{\sec^{-1}(3x^2)}]$

$y = \tan z$ $\frac{opp}{adj}$

$y = \frac{\sqrt{9x^4-1}}{1} = \boxed{\sqrt{9x^4-1}}$

Example:

Let $x = \csc^{-1}\sqrt{5x}$

Evaluate the following: $y = \cos\left[\underline{\csc^{-1}\sqrt{5x}}\right]$

① $x = \underline{\csc^{-1}\sqrt{5x}}$

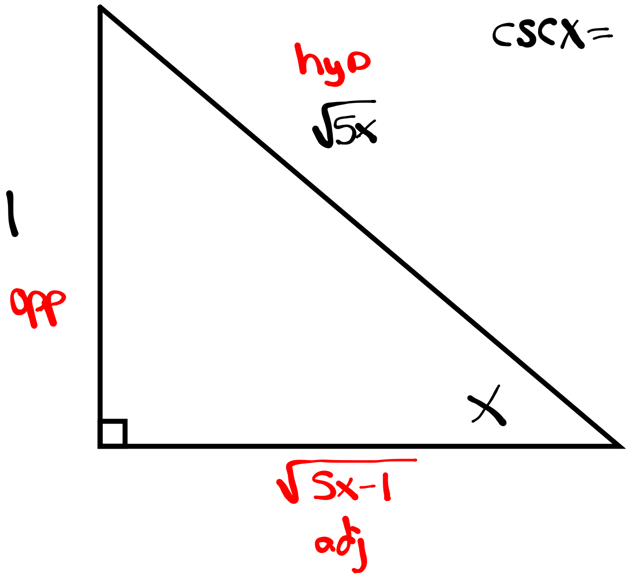
② $y = \cos\left[\underline{\csc^{-1}\sqrt{5x}}\right]$

$\csc x = \frac{\sqrt{5x}}{1}$ $\frac{hyp}{opp}$

$y = \cos x$

$y = \frac{\sqrt{5x-1}}{\sqrt{5x}}$

$y = \frac{\sqrt{25x^2-5x}}{5x}$



$a^2+b^2=c^2$

$1^2+b^2=(\sqrt{5x})^2$

$1+b^2=5x$

$b^2=5x-1$

$b=\sqrt{5x-1}$

<u>Review</u> of Trig Derivatives from last month

In a nutshell....we have

$$\frac{d}{du}\left(\sin u\right)=\cos u \bullet du$$

$$\frac{d}{du}\left(\csc u\right)=-\csc u\cot u \bullet du$$

$$\frac{d}{du}\left(\cos u\right)=-\sin u \bullet du$$

$$\frac{d}{du}\left(\sec u\right)=\sec u\tan u \bullet du$$

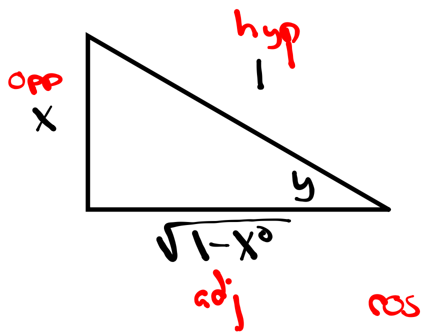
$$\frac{d}{du}\left(\tan u\right)=\sec^2 u \bullet du$$

$$\frac{d}{du}\left(\cot u\right)=-\csc^2 u \bullet du$$

Differentiating Inverse Trigonometric Functions

Let's start with finding the derivative of $y = \arcsin x$ ($y = \sin^{-1} x$) which is... $\sin y = x$

$$\boxed{\sin y = x}$$



$$(\cos y)\frac{dy}{dx} = 1$$

$$\cos y = \frac{\sqrt{1-x^2}}{1}$$

$$\frac{dy}{dx} = \frac{1}{\boxed{\cos y}}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \boxed{\frac{1}{\sqrt{1-x^2}}}$$

Once again like before, if $y = \sin^{-1} u$, where u is a differentiable function of x, then application of the chain rule gives us:

$$\boxed{\frac{dy}{dx}\left(\sin^{-1} u\right) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}}$$

In a similar manner, the derivative of $y = \arccos u$ ($y = \cos^{-1} u$) can be shown to be:

$$\boxed{\frac{dy}{dx}\left(\cos^{-1} u\right) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}}$$

We list the derivatives of the six inverse trigonometric functions...

(arcsin, arccos, arctan, arccsc, arcsec, arccot)

In each case *u* denotes a differentiable function of x.

$$\boxed{\frac{d}{dx}\left(\arcsin u\right)=\frac{1}{\sqrt{1-u^{2}}}\frac{du}{dx}}$$

$$\boxed{\frac{d}{dx}\left(\arccos u\right)=\frac{-1}{\sqrt{1-u^{2}}}\frac{du}{dx}}$$

$$\boxed{\frac{d}{dx}\left(\arctan u\right)=\frac{1}{1+u^{2}}\frac{du}{dx}}$$

$$\boxed{\frac{d}{dx}\left(arc\csc u\right)=\frac{-1}{|u|\sqrt{u^{2}-1}}\frac{du}{dx}}$$

$$\boxed{\frac{d}{dx}\left(arc\sec u\right)=\frac{1}{|u|\sqrt{u^{2}-1}}\frac{du}{dx}}$$

$$\boxed{\frac{d}{dx}\left(arc\cot u\right)=\frac{-1}{1+u^{2}}\frac{du}{dx}}$$

There is no common agreement on the definition of arcsec x (or arccsc x) for negative values of x. When we defined the range of arcsec x, we chose to preserve the identity arcsec x = arccos (1/x). One of the consequences of this choice is that the slope of the graph of the inverse secant function is always positive, which accounts for the absolute value sign in the formula for the derivative of arcsec x.

Examples:

Differentiate each of the following and simplify your answers

$y=\cos^{-1}(2x^2)$ $u=2x^2$ $du=4x$

$$\frac{dy}{dx}=\frac{-1}{\sqrt{1-(2x^2)^2}}\cdot 4x$$

$$\frac{dy}{dx}=\frac{-4x}{\sqrt{1-4x^4}}$$

$y=\arctan 3x$ $u=3x$ $du=3$

$y=\tan^{-1}(3x)$

$$\frac{dy}{dx}=\frac{1}{1+(3x)^2}\cdot 3$$

$$\frac{dy}{dx}=\frac{3}{1+9x^2}$$

$y=\sin^{-1}\sqrt{x}$ $u=\sqrt{x}=x^{1/2}$ $du=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-(\sqrt{x})^2}}\cdot\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{x}\sqrt{1-x}}=\frac{1}{\sqrt{4x(1-x)}}=\frac{1}{\sqrt{4x-4x^2}}$$

$y=2\arccos 2x+\sqrt{1-4x^2}$ $u=2x$ $du=2$ $\longrightarrow (1-4x^2)^{1/2}$

$$\frac{dy}{dx}=2\left[\frac{-1}{\sqrt{1-(2x)^2}}\cdot 2\right]+\frac{1}{\cancel{2}}(1-4x^2)^{-1/2}\left(\overset{-4x}{\cancel{-8x}}\right)$$

$$\frac{dy}{dx}=\frac{-4}{\sqrt{1-4x^2}}-\frac{4x}{\sqrt{1-4x^2}}=\boxed{\frac{-4-4x}{\sqrt{1-4x^2}}}$$

$f(x)=(x)(\tan^{-1}\sqrt{x})$ $u=\sqrt{x}$ $du=\frac{1}{2\sqrt{x}}$

$$f'(x)=1(\tan^{-1}\sqrt{x})+x\left[\frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}}\right]$$

$$f'(x)=\tan^{-1}\sqrt{x}+\frac{x}{2\sqrt{x}(1+x)}$$ $\longrightarrow \frac{x}{x^{1/2}}=x^{1-1/2}=x^{1/2}$

$$\boxed{f'(x)=\tan^{-1}\sqrt{x}+\frac{\sqrt{x}}{2(1+x)}}$$

Examples:

Differentiate each of the following...

$u=3x^2 \quad du=6x$

$$f(x)=(x^3)(\sin^{-1}(3x^2))$$

$$f'(x)=3x^2\sin^{-1}(3x^2)+x^3\left[\frac{1}{\sqrt{1-(3x^2)^2}}\cdot 6x\right]$$

$$f'(x)=3x^2\sin^{-1}(3x^2)+\frac{6x^4}{\sqrt{1-9x^4}}$$

$$f(x)=\sqrt{3x-\tan^{-1}\sqrt{x}} = (3x-\tan^{-1}\sqrt{x})^{1/2}$$

$u=\sqrt{x}$

$du=\frac{1}{2\sqrt{x}}$

$$f'(x)=\frac{1}{2}(3x-\tan^{-1}\sqrt{x})^{-1/2}\left(3-\left(\frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}}\right)\right)$$

$$f'(x)=\left(\frac{1}{2\sqrt{3x-\tan^{-1}\sqrt{x}}}\right)\left(3-\frac{1}{2\sqrt{x}(1+x)}\right)$$

Homework:

Page 339
#1, 2, 4, 6

Questions from Homework

Page 313

① l) $y = \dfrac{\sin x}{1+\cos x}$

$$y' = \frac{\cos x\,(1+\cos x) - \sin x\,(-\sin x)}{(1+\cos x)^2}$$

$$y' = \frac{\cos x + \underline{\cos^2 x + \sin^2 x}}{(1+\cos x)^2}$$

$$y' = \frac{1 + \cos x}{(1+\cos x)^2}$$

$$y' = \frac{1}{1+\cos x}$$

m) $y = (1+\cos^2 x)^6$

$u = (\cos x)^2$

$du = 2(\cos x)(-\sin x)$

$du = -2\sin x \cos x$

$$\frac{dy}{dx} = 6(1+\cos^2 x)^5(-\underline{2\sin x \cos x})$$

$$\frac{dy}{dx} = -6\,\underline{\sin 2x}\,(1+\cos^2 x)^5$$

Questions from Homework

Page 313

③ Find an equation of the tangent line to the curve at the given point.

a) $y = 2\sin x$ at $(\frac{\pi}{6}, 1)$

(i) Find the derivative:

$y = 2\sin x$

$y' = 2\cos x$

(ii) Find m:

$y' = 2\cos x$

$y' = 2\cos(\frac{\pi}{6})$

$y' = 2\left(\frac{\sqrt{3}}{2}\right)$

$y' = \sqrt{3}$

$m = \sqrt{3}$

(iii) Find equation:

$y - y_1 = m(x - x_1)$

$y - 1 = \sqrt{3}(x - \frac{\pi}{6})$

$y - 1 = x\sqrt{3} - \frac{\pi\sqrt{3}}{6}$

$y = x\sqrt{3} - \frac{\pi\sqrt{3}}{6} + 1$

$0 = x\sqrt{3} - y - \frac{\pi\sqrt{3}}{6} + 1$

$$\boxed{0 = 6x\sqrt{3} - 6y - \pi\sqrt{3} + 6}$$

Questions from Homework

$u = \frac{1}{2}x^3 \quad du = \frac{3x^2}{2}$

① e) $y = \cos^{-1}\left(\frac{x^3}{2}\right)$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{x^3}{2}\right)^2}} \cdot \frac{3x^2}{2}$$

$$\frac{dy}{dx} = \frac{-3x^2}{\underline{2}\sqrt{1-\frac{x^6}{4}}}$$

$$\frac{dy}{dx} = \frac{-3x^2}{\sqrt{4\left(1-\frac{x^6}{4}\right)}}$$

$$\frac{dy}{dx} = \frac{-3x^2}{\sqrt{4-x^6}}$$

Questions from Homework

① i) $y = \sin^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

$u = \frac{\cos x}{1+\sin x}$

$du = \frac{-\sin x(1+\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$

$du = \frac{-\sin x - \sin^2 - \cos^2 x}{(1+\sin x)^2}$

$du = \frac{-(\sin x + \underline{\sin^2 x + \cos^2 x})}{(1+\sin x)^2}$

$du = \frac{-(\cancel{1+\sin x})}{\cancel{(1+\sin x)^2}}$

$du = \frac{-1}{1+\sin x}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{\cos x}{1+\sin x}\right)^2}} \cdot \frac{-1}{1+\sin x}$$

$$\frac{dy}{dx} = \frac{-1}{\underline{(1+\sin x)}\sqrt{1-\frac{(\cos x)^2}{(1+\sin x)^2}}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{\underline{(1+\sin x)^2}\left(1-\frac{\cos^2 x}{(1+\sin x)^2}\right.}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{\underline{(1+\sin x)^2} - \cos^2 x}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{\underline{1+2\sin x+\sin^2 x} - \underset{(1-\sin^2 x)}{\cos^2 x}}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{\cancel{1}+2\sin x+\underline{\sin^2 x}-\cancel{1}+\underline{\sin^2 x}}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{2\sin x+2\sin^2 x}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{2\sin x(1+\sin x)}}$$

Questions from Homework

② $f(x) = (x)(\tan^{-1}x)$ at $x = 1$

$$f'(x) = 1\tan^{-1}x + x\left(\frac{1}{1+x^2}\right)(1)$$

$$f'(x) = \tan^{-1}x + \frac{x}{1+x^2}$$

$$f'(1) = \tan^{-1}(1) + \frac{(1)}{1+(1)^2}$$

what angle has a tangent value equal to 1

$$f'(1) = \frac{\pi}{4} + \frac{1}{2}$$

$$f'(1) = \frac{\pi}{4} + \frac{2}{4} = \boxed{\frac{\pi+2}{4}}$$

④ $f(x) = (3\tan^{-1}x)^4$

$$f'(x) = 4(3\tan^{-1}x)^3\left[\cancel{(0)(\tan^{-1}x)} + 3\left(\frac{1}{1+x^2}\right)(1)\right]$$

$$f'(x) = 4(3\tan^{-1}x)^3\left(\frac{3}{1+x^2}\right) = \frac{12(3\tan^{-1}x)^3}{1+x^2}$$

$$f'(\sqrt{3}) = \frac{12(3\tan^{-1}(\sqrt{3}))^3}{1+(\sqrt{3})^2}$$

what angle has a tangent value equal to $\sqrt{3}$

$$f'(\sqrt{3}) = \frac{12\left(3\left(\frac{\pi}{3}\right)\right)^3}{1+3}$$

$$f'(\sqrt{3}) = \frac{12\pi^3}{4} = \boxed{3\pi^3}$$

⑥ $f(x)=(x-3)(6x-x^2)^{1/2}+9\sin^{-1}\left(\frac{x-3}{3}\right)$

$f'(x)=1(6x-x^2)^{1/2}+(x-3)\left(\frac{1}{2}\right)(6x-x^2)^{-1/2}(6-2x)+9\left(\frac{1}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}}\right)\left(\frac{1}{3}\right)$

$f'(x)=\sqrt{6x-x^2}+\frac{(x-3)(3-x)}{\sqrt{6x-x^2}}+\frac{3}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}}$

$f'(3)=3+0+3=\boxed{6}$