## Questions from Homework

Exercise 2.8

(4) If $f(x) = \sqrt{1+x^3}$ find $f''(2)$

$$f(x) = (1+x^3)^{1/2}$$

$$f'(x) = \frac{1}{2}(1+x^3)^{-1/2}(3x^2)$$

$$f'(x) = \frac{3x^2}{2(1+x^3)^{1/2}}$$

$$f''(x) = \frac{\overset{f'}{6x}\left(\overset{g}{2(1+x^3)^{1/2}}\right) - \overset{f}{3x^2}\,\overset{g'}{(1+x^3)^{-1/2}(3x^2)}}{4(1+x^3)}$$

$$f''(x) = \frac{12x(1+x^3)^{1/2} - 9x^4(1+x^3)^{-1/2}}{4(1+x^3)}$$

$$4 + 4x^3 - 3x^3$$

$$f''(x) = \frac{3x(1+x^3)^{-1/2}\left[4(1+x^3) - 3x^3\right]}{4(1+x^3)}$$

$$f''(x) = \frac{3x(4+x^3)}{4(1+x^3)^{3/2}}$$

$$f''(2) = \frac{3(2)(4+(2)^3)}{4(1+(2)^3)^{3/2}}$$

$$f''(2) = \frac{72}{108}$$

$$\boxed{f''(2) = \frac{2}{3}}$$

## Questions from Homework

(8) Find a quadratic function $f$ such that

$f(3) = 33$

$f'(3) = 22$

$f''(3) = 8$

$$\boxed{f(x) = 4x^2 - 2x + 3}$$

$f'(x) = 8x - 2$

$f''(x) = 8$

# Differentiation Rules

## Product Rule:

**The Product Rule** If $f$ and $g$ are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Express the product rule verbally if you are considering a function of the form...

$f(x)$= (First) x (Second)

" The derivative of the product of two functions is the the first multiplied by the derivative of second, plus the derivative of first multiplied by the second"

*Get in the habit of verbalizing the rule as you differentiate...it will help when the functions get more complicated.*

# Quotient Rule:

**The Quotient Rule** If $f$ and $g$ are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Express the quotient rule verbally ...

" The denominator multiplied by the derivative of the numerator, minus the numerator multiplied by the derivative of the denominator, all over the denominator squared"

# Combining the Chain Rule With the Product and Quotient Rule:

**The Chain Rule** If $f$ and $g$ are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then $F$ is differentiable and $F'$ is given by the product

$$F'(x) = f'(g(x))g'(x)$$

Differentiate the following function and simplify your answer:

$$y = (x^2+1)^3(2-3x)^4$$

$$y' = (x^2+1)^3(4)(2-3x)^3(-3) + 3(x^2+1)^2(2x)(2-3x)^4$$

$$= -12(x^2+1)^3(2-3x)^3 + 6x(x^2+1)^2(2-3x)^4$$

$$= -6(x^2+1)^2(2-3x)^3\left[2(x^2+1) - x(2-3x)\right]$$

$$= -6(x^2+1)^2(2-3x)^3\left[2x^2+2-2x+3x^2\right]$$

$$= \boxed{-6(x^2+1)^2(2-3x)^3(5x^2-2x+2)}$$

$$g(x) = \frac{(3x+2)^2}{2x}$$

$$g'(x) = \frac{2x(2)(3x+2)(3) - (3x+2)^2(2)}{(2x)^2}$$

$$= \frac{12x(3x+2) - 2(3x+2)^2}{4x^2}$$

$$= \frac{2(3x+2)\left[6x - (3x+2)\right]}{4x^2}$$

$$= \frac{\cancel{2}(3x+2)(3x-2)}{\cancel{4}x^2}$$

$$= \boxed{\frac{(3x+2)(3x-2)}{2x^2}} \quad \text{or} \quad \frac{9x^2-4}{2x^2}$$

Differentiate the following functions and simplify your answers:

$$s=\left(\frac{2t-1}{t+2}\right)^6$$

$$2t+4-2t+1$$

$$\frac{ds}{dt}=6\left[\frac{2t-1}{t+2}\right]^5\left[\frac{(t+2)(2)-(2t-1)(1)}{(t+2)^2}\right]$$

$$\frac{ds}{dt}=6\left[\frac{(2t-1)^5}{(t+2)^5}\right]\left[\frac{5}{(t+2)^2}\right]$$

$$\frac{ds}{dt}=\frac{30(2t-1)^5}{(t+2)^7}$$

$$g(x)=(9x^{-3})(5x^3-1)^6$$

$$g'(x)=(9x^{-3})[6(5x^3-1)^5(15x^2)]-27x^{-4}(5x^3-1)^6$$

$$g'(x)=810x^{-1}(5x^3-1)^5-27x^{-4}(5x^3-1)^6$$

$$30x^3-5x^3+1$$

$$g'(x)=27x^{-4}(5x^3-1)^5\left[30x^3-(5x^3-1)\right]$$

$$g'(x)=27x^{-4}(5x^3-1)^5(25x^3+1)$$

$$g'(x)=\frac{27(5x^3-1)^5(25x^3+1)}{x^4}$$

$F(x) = f(x)g(x)$ product $F(x) = (f \cdot g)(x)$ $f'(x)g(x) + f(x)g'(x)$

$F(x) = \frac{f(x)}{g(x)}$ quotient $F(x) = \left(\frac{f}{g}\right)(x)$ $\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

$F(x) = f(g(x))$ chain $F(x) = (f \circ g)(x)$ $f'(g(x))g'(x)$

## Example 1

Chain Rule

Let $F(x) = f(g(x))$

If $f(2) = 3$, $f'(2) = 5$, $g(1) = 2$ and $g'(1) = 4$ find $F'(1)$.

$F'(x) = f'(g(x))g'(x)$

$F'(1) = f'(g(1))g'(1)$

$F'(1) = f'(2) \cdot (4)$

$F'(1) = (5) \cdot (4)$

$\boxed{F'(1) = 20}$

# Example 2

If $y = u^{10} + u^5 + 2$, where $u = 1 - 3x^2$, find $\left.\frac{dy}{dx}\right]_{x=1}$

① $\frac{dy}{du} = 10u^9 + 5u^4$

② $\frac{du}{dx} = -6x$

③ when $x = 1$

$u = 1 - 3x^2$

$u = 1 - 3(1)^2$

$u = -2$

④ $\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$

$\left.\frac{dy}{dx}\right]_{x=1} = [10u^9 + 5u^4][-6x]$

$\left.\frac{dy}{dx}\right]_{x=1} = [10(-2)^9 + 5(-2)^4][-6(1)]$

$\left.\frac{dy}{dx}\right]_{x=1} = [-5120 + 80][-6]$

$\left.\frac{dy}{dx}\right]_{x=1} = [-5040][-6]$

$\left.\frac{dy}{dx}\right]_{x=1} = 30240$

# Homework

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④ e) $y = \sqrt{x}(5-\sqrt{x}) = 5\sqrt{x} - x = 5x^{1/2} - x$

$y' = \frac{5}{2}x^{-1/2} - 1$

$y' = \frac{5}{2\sqrt{x}} - 1 = \frac{5-2\sqrt{x}}{2\sqrt{x}}$

Using Product:

$y = \sqrt{x}(5-\sqrt{x})$ $\qquad y' = \frac{1}{2}x^{-1/2}(5-\sqrt{x}) + x^{1/2}\left(-\frac{1}{2}x^{-1/2}\right)$

$y' = \frac{1}{2\sqrt{x}}(5-\sqrt{x}) + \sqrt{x}\left(-\frac{1}{2\sqrt{x}}\right)$

$y' = \frac{5}{2\sqrt{x}} - \frac{\sqrt{x}}{2\sqrt{x}} - \frac{\sqrt{x}}{2\sqrt{x}}$

$y' = \frac{5}{2\sqrt{x}} - \frac{2\sqrt{x}}{2\sqrt{x}}$

$y' = \frac{5}{2\sqrt{x}} - 1$

f) $y = \frac{x^2 - 2x}{\sqrt{x}}$

$y' = \frac{x^{1/2}(2x-2) - \frac{1}{2}x^{-1/2}(x^2-2x)}{x}$

$y' = \frac{2x^{3/2} \cdot 2x^{1/2} - 2x^{1/2} \cdot 2x^{1/2} - \frac{1}{\cancel{2\sqrt{x}}}(x^2-2x) \cdot \cancel{2x^{1/2}}}{x \cdot 2x^{1/2}}$

$y' = \frac{4x^2 - 4x - x^2 + 2x}{2x^{3/2}} = \frac{3x^2 - 2x}{2x^{3/2}}$

$= \frac{x(3x-2)}{2x^{3/2}}$

$= \frac{x^{-1/2}(3x-2)}{2}$

$= \boxed{\frac{3x-2}{2\sqrt{x}}}$

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⑥ if $y = u^2 - u^3 + 2u^4$ and $u = \frac{x}{2x-1}$ Find $\left.\frac{dy}{dx}\right|_{x=1}$

(i) $\frac{dy}{du} = 2u - 3u^2 + 8u^3$

(ii) $\frac{du}{dx} = \frac{1(2x-1) - 2x}{(2x-1)^2}$ $\quad \cancel{2x} - 1 - \cancel{2x}$

$\frac{du}{dx} = \frac{-1}{(2x-1)^2}$

(iii) Find u

$u = \frac{x}{2x-1}$

$u = \frac{(1)}{2(1)-1}$

$u = 1$

(iv) $\left.\frac{dy}{dx}\right|_{x=1} = \left[\frac{dy}{du}\right] \cdot \left[\frac{du}{dx}\right]$

$\left.\frac{dy}{dx}\right|_{x=1} = \left[2u - 3u^2 + 8u^3\right]\left[\frac{-1}{(2x-1)^2}\right]$

$\left.\frac{dy}{dx}\right|_{x=1} = \left[2(1) - 3(1)^2 + 8(1)^3\right]\left[\frac{-1}{(2(1)-1)^2}\right]$

$\left.\frac{dy}{dx}\right|_{x=1} = [2 - 3 + 8][-1] = -7$

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9) d, $y = x\sqrt{x^2+5}$ , $(-2,-6)$

$x_1 = -2$

$y_1 = -6$

① Find $y'$

$$y' = 1\sqrt{x^2+5} + x\left[\frac{1}{\cancel{2}}(x^2+5)^{-1/2}\cdot \cancel{2}x\right]$$

$$y' = \sqrt{x^2+5} + \frac{x^2}{\sqrt{x^2+5}}$$

② Find m or $y'(-2)$

$$y'(-2) = \sqrt{(-2)^2+5} + \frac{(-2)^2}{\sqrt{(-2)^2+5}}$$

$$y' = 3 + \frac{4}{3} = \boxed{\frac{13}{3}}$$

③ Find equation:

$$y+6 = \frac{13}{3}(x+2)$$

$$y+6 = \frac{13x}{3} + \frac{26}{3} - 6$$

$$\boxed{y = \frac{13x}{3} + \frac{8}{3}}$$

$$3y = 13x + 8$$

$$\boxed{0 = 13x - 3y + 8}$$

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(12) Find the points on the curve $y=\frac{1}{2x-1}$ where the tangent line is perpendicular to $x-2y=1$

(I) Find perpendicular slope:

$x-2y=1$

$-2y=-x+1$

$y=\frac{-x}{-2}+\frac{1}{-2}$

$y=\frac{1}{2}x-\frac{1}{2}$

$m=\frac{1}{2}$

$\boxed{m\perp=-2}$

(II) Find derivative:

$y=\frac{1}{2x-1}=(2x-1)^{-1}$

$y'=-1(2x-1)^{-2}(2)$

$\boxed{y'=\frac{-2}{(2x-1)^2}}$

(III) Solve for x

$\frac{-2}{1}=\frac{-2}{(2x-1)^2}$

$-2(2x-1)^2=-2$

$(2x-1)^2=1$

$4x^2-4x+1=1$

$4x^2-4x=0$

$4x(x-1)=0$

$4x=0 \quad | \quad x-1=0$

$x=0 \quad | \quad x=1$

(IV) Solve for y:

if $x=0$

$y=\frac{1}{2x-1}$

$y=\frac{1}{-1}=-1$

$\boxed{(0,-1)}$

if $x=1$

$y=\frac{1}{2x-1}$

$y=\frac{1}{1}=1$

$\boxed{(1,1)}$

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(3) Find $\left.\frac{dy}{dx}\right]_{x=4}$ if $y=u^2-2u^5$ and $u=x-\sqrt{x}$

(1) Find $u$

$u=x-\sqrt{x}$

$u=(4)-\sqrt{4}$

$\boxed{u=2}$

(2) Find $\frac{dy}{du}$

$y=u^2-2u^5$

$\frac{dy}{du}=2u-10u^4$

(3) Find $\frac{du}{dx}$

$u=x-x^{1/2}$

$\frac{du}{dx}=1-\frac{1}{2}x^{-1/2}$

$$\left.\frac{dy}{dx}\right]_{x=4}=(2u-10u^4)\left(1-\frac{1}{2\sqrt{x}}\right)$$

$$=(2(2)-10(2)^4)\left(1-\frac{1}{2\sqrt{4}}\right)$$

$$=(4-160)\left(1-\frac{1}{4}\right)$$

$$=(-156)\left(\frac{3}{4}\right)$$

$$=\frac{-468}{4}$$

$$=-117$$

(9) If $F(x)=f(g(x))$, where $\boxed{g(2)=4}$, $\boxed{g'(2)=3}$, $\boxed{f'(4)=5}$ find $F'(2)$

$$F'(x)=f'(g(x))\cdot g'(x)$$

$$F'(2)=f'(\underline{g(2)})\cdot g'(2)$$

$$F'(2)=\boxed{f'(4)}\cdot\boxed{g'(2)}$$

$$F'(2)=5\cdot3=\underline{15}$$

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#3-10

④ $\frac{dy}{dt}\Big]_{t=1}$

$y=\sqrt{1+r^2}$

$\frac{dy}{dr}=\frac{1}{2}(1+r^2)^{-1/2}(2r)$

$\frac{dy}{dr}=\frac{r}{\sqrt{1+r^2}}$

$r=\frac{t+1}{2t+1}$

$\frac{dr}{dt}=\frac{1(2t+1)-2(t+1)}{(2t+1)^2}$

$\frac{dr}{dt}=\frac{-1}{(2t+1)^2}$

when $t=1$ $r=\frac{2}{3}$

$$\frac{dy}{dt}\Big]_{t=1}=\left[\frac{dy}{dr}\right]\left[\frac{dr}{dt}\right]$$

$$=\left[\frac{r}{\sqrt{1+r^2}}\right]\left[\frac{-1}{(2t+1)^2}\right]$$

$$=\left[\frac{2/3}{\sqrt{1+(2/3)^2}}\right]\left[\frac{-1}{(2(1)+1)^2}\right]$$

$$=\left[\frac{2/3}{\frac{\sqrt{13}}{\sqrt{9}}}\right]\left[-\frac{1}{9}\right]$$

$$=\left[\frac{2}{\cancel{3}}\cdot\frac{\cancel{3}}{\sqrt{13}}\right]\left[-\frac{1}{9}\right]$$

$$=\boxed{\frac{-2}{9\sqrt{13}}}$$

If $f(3) = -2$, $f'(3) = 3$, $g(3) = 1$, $g'(3) = 7$

and $f'(1) = 4$

Find:

(i) $(f \cdot g)'(3) = f'(3)g(3) + f(3) \cdot g'(3)$

$= (3)(1) + (-2)(7)$

$= 3 - 14$

$= -11$

(ii) $\left(\frac{f}{g}\right)'(3) = \frac{f'(3)g(3) - g'(3)f(3)}{[g(3)]^2}$

$= \frac{(3)(1) - (7)(-2)}{(1)^2}$

$= \frac{3+14}{1}$

$= 17$

(iii) $(f \circ g)'(3) = f'(g(3)) \cdot g'(3)$

$= f'(1) \cdot g'(3)$

$= 4 \cdot 7$

$= 28$