1.4 Proving Conjectures: Deductive Reasoning

GOAL

Prove mathematical statements using a logical argument.

EXPLORE...

- How can the conjecture "All teens like music" be supported inductively? Can this conjecture be proved? Explain.

SAMPLE ANSWER

The conjecture "All teens like music" can be supported inductively by collecting more evidence. A questionnaire or an online survey could be tools to help gather the evidence. The conjecture cannot be proved because it is impossible to ask all teens. However, the conjecture can be refuted with one counterexample: a student who dislikes music.

LEARN ABOUT the Math

Jon discovered a pattern when adding integers:

$$1 + 2 + 3 + 4 + 5 = 15$$
$$(-15) + (-14) + (-13) + (-12) + (-11) = -65$$
$$(-3) + (-2) + (-1) + 0 + 1 = -5$$

3(5) = 15
-13(5) = -65
add
in a row

He claims that whenever you add five consecutive integers, the sum is always 5 times the median of the numbers.

middle

❓ How can you prove that Jon's conjecture is true for all integers?

EXAMPLE 1 Connecting conjectures with reasoning

Prove that Jon's conjecture is true for all integers.

Pat's Solution

$$5(3) = 15$$
$$5(-13) = -65$$
$$5(-1) = -5$$

The median is the middle number in a set of integers when the integers are arranged in consecutive order. I observed that Jon's conjecture was true in each of his examples.

(inductive)

$$210 + 211 + 212 + 213 + 214 = 1060$$
$$5(212) = 1060$$

I tried a sample with greater integers, and the conjecture still worked.

Let x represent any integer.
Let S represent the sum of five consecutive integers.

$$S = \underbrace{(x - 2) + (x - 1) + x + (x + 1) + (x + 2)}_{\text{5 consecutive integers}}$$

$$S = x - 2 + x - 1 + x + x + 1 + x + 2$$

I decided to start my **proof** by representing the sum of five consecutive integers. I chose x as the median and then wrote a **generalization** for the sum.

proof
A mathematical argument showing that a statement is valid in all cases, or that no counterexample exists.

generalization
A principle, statement, or idea that has general application.

$$S = (x + x + x + x + x) + (-2 + (-1) + 0 + 1 + 2)$$
$$S = 5x + 0$$

I simplified by gathering like terms.

$$\boxed{S = 5x}$$

Jon's conjecture is true for all integers.

Since x represents the median of five consecutive integers, $5x$ will always represent the sum.

Let x = any interger

Let S = sum of 5 consecutive

$$S = x + (x+1) + (x+2) + (x+3) + (x+4)$$

$$S = \underline{x} + \underline{x} + \underline{1} + \underline{x} + \underline{2} + \underline{x} + \underline{3} + \underline{x} + \underline{4}$$

$$S = 5x + 10$$

Reflecting

A. What type of reasoning did Jon use to make his conjecture?

B. Pat used **deductive reasoning** to prove Jon's conjecture. How does this differ from the type of reasoning that Jon used?

✱ deductive reasoning

Drawing a specific conclusion through logical reasoning by starting with general assumptions that are known to be valid.

Answers

specific

A. Jon used inductive reasoning to make his conjecture. He analyzed a pattern he noticed and developed a conjecture about this pattern.

B. Pat's reasoning differed from Jon's because she represented any five consecutive integers with variables, not with specific sets of five consecutive integers as Jon did. Because Pat's deductive reasoning showed that the conjecture was true for any five consecutive integers, she proved that the conjecture was true for all cases. Jon was only able to say that the conjecture was true for the specific sets of consecutive integers that he sampled.

- (using algebra to prove)
- variables

APPLY the Math

EXAMPLE 2 Using deductive reasoning to generalize a conjecture

subtract in a row

In Lesson 1.3, page 19, Luke found more support for Steffan's conjecture from Lesson 1.1, page 9—that the difference between consecutive perfect squares is always an odd number.

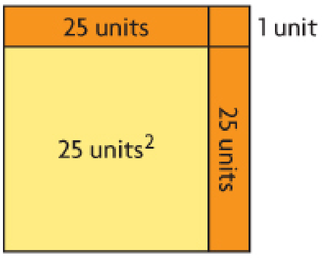
exponent of 2

Determine the general case to prove Steffan's conjecture.

Gord's Solution

The difference between consecutive perfect squares is always an odd number.

Steffan's conjecture has worked for consecutive perfect squares with sides of 1 to 7 units.



$26^2 - 25^2 = 2(25) + 1$
$26^2 - 25^2 = 51$

I tried a sample using even greater squares: 26^2 and 25^2.

The difference is the two sets of 25 unit tiles, plus a single unit tile.

Let x be any natural number.
Let D be the difference between consecutive perfect squares.
$D = (x + 1)^2 - x^2$

Since the conjecture has been supported with specific examples, I decided to express the conjecture as a general statement. I chose x to be the length of the smaller square's sides. The larger square's sides would then be $x + 1$.

$D = x^2 + x + x + 1 - x^2$
$D = x^2 + 2x + 1 - x^2$
$\boxed{D = 2x + 1}$

I expanded and simplified my expression. Since x represents any natural number, $2x$ is an even number, and $2x + 1$ is an odd number.

Steffan's conjecture, that the difference of consecutive perfect squares is always an odd number, has been proved for all natural numbers.

there are no counterexamples

Let x = any integer

Let D = the difference between consecutive perfect squares.

$D = (x+1)^2 - x^2$ or $D = x^2 - (x-1)^2$

$D = (x+1)(x+1) - x^2$

$D = \cancel{x^2} + \underline{\underline{x}} + \underline{\underline{x}} + 1 - \cancel{x^2}$

$\boxed{D = 2x+1}$

$D = x^2 - (x-1)(x-1)$

$D = x^2 - (x^2 - x - x + 1)$

$D = x^2 - (x^2 - 2x + 1)$

$D = \cancel{x^2} - \cancel{x^2} + 2x - 1$

$\boxed{D = 2x-1}$

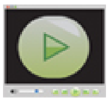
EXAMPLE 2 Using deductive reasoning to generalize a conjecture

In Lesson 1.3, page 19, Luke found more support for Steffan's conjecture from Lesson 1.1, page 9—that the difference between consecutive perfect squares is always an odd number.

Determine the general case to prove Steffan's conjecture.

Your Turn

In Lesson 1.3, Luke visualized the generalization but did not develop the reasoning to support it. How did the visualization explained by Luke help Gord develop the general statement? Explain.



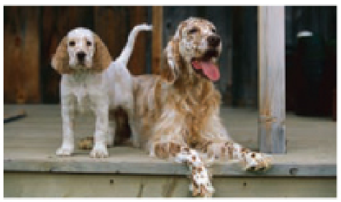
Answer

Luke's visualization may have helped Gord understand that the difference is always going to have two equal sets of tiles, plus one more. Since two equal sets will always represent an even number ($2n$ is an even number), the additional single tile will always make the difference odd.

EXAMPLE 3 Using deductive reasoning to make a valid conclusion

All dogs are mammals. All mammals are vertebrates. Shaggy is a dog.

What can be deduced about Shaggy?



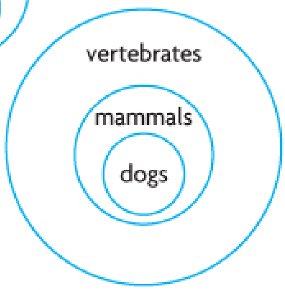
Oscar's Solution

Shaggy is a dog.

All dogs are mammals.

These statements are given. I represented them using a Venn diagram.

All mammals are vertebrates.



This statement is given. I modified my diagram.

Therefore, through deductive reasoning, Shaggy is a mammal and a vertebrate.

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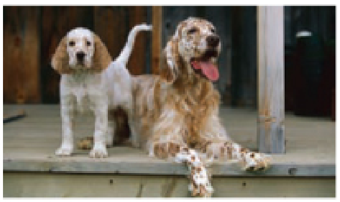
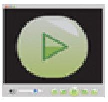


Your Turn

Weight-lifting builds muscle. Muscle makes you strong. Strength improves balance. Inez lifts weights. What can be deduced about Inez?

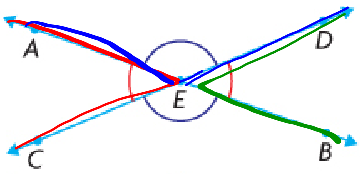


Answer

I can deduce that Inez is building muscle. The other connections from the given statements lead from weight-lifting, but I cannot deduce that Inez is either strong or has improved balance. The act of building muscle does not mean that you have currently gained the muscle needed for strength and improved balance.

EXAMPLE 4 Using deductive reasoning to prove a geometric conjecture

Prove that when two straight lines intersect, the vertically opposite angles are equal.



Jose's Solution: Reasoning in a two-column proof

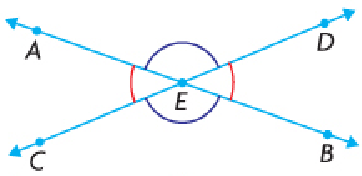
Statement	Justification
$\angle AEC + \angle AED = 180°$	Supplementary angles
$\angle AEC = 180° - \angle AED$	Subtraction property
$\angle BED + \angle AED = 180°$	Supplementary angles
$\angle BED = 180° - \angle AED$	Subtraction property
$\angle AEC = \angle BED$	**Transitive property**

transitive property

If two quantities are equal to the same quantity, then they are equal to each other.
If $a = b$ and $b = c$, then $a = c$.

EXAMPLE 4 Using deductive reasoning to prove a geometric conjecture

Prove that when two straight lines intersect, the vertically opposite angles are equal.



Your Turn

Use a **two-column proof** to prove that $\angle AED$ and $\angle CEB$ are equal.

two-column proof
A presentation of a logical argument involving deductive reasoning in which the statements of the argument are written in one column and the justifications for the statements are written in the other column.

Answer

Statement	Justification	Explanation
$\angle AEC + \angle AED = 180°$	Supplementary angles	The measures of two angles that lie on the same straight line have a sum of 180°.
$\angle AED = 180° - \angle AEC$	Subtraction property	
$\angle CEB + \angle AEC = 180°$	Supplementary angles	
$\angle CEB = 180° - \angle AEC$	Subtraction property	
$\angle AED = \angle CEB$	Transitive property	Two quantities that are equal to the same quantity are equal to each other. In this example, both angle measures are equal to $180° - \angle AEC$.

In Summary

Key Idea

- Deductive reasoning involves starting with general assumptions that are known to be true and, through logical reasoning, arriving at a specific conclusion.

Need to Know

- A conjecture has been proved only when it has been shown to be true for every possible case or example. This is accomplished by creating a proof that involves general cases.
- When you apply the principles of deductive reasoning correctly, you can be sure that the conclusion you draw is valid.
- The transitive property is often useful in deductive reasoning. It can be stated as follows: Things that are equal to the same thing are equal to each other. If $a = b$ and $b = c$, then $a = c$.
- A demonstration using an example is *not* a proof.

Assignment: pages 31-33

Questions: 2, 4, 5, 6, 7,10,16,17

SOLUTIONS => 1.4 Proving Conjectures: Deductive Reasoning

2. Jim is a barber. Everyone whose hair is cut by Jim gets a good haircut. Austin's hair was cut by Jim. What can you deduce about Austin?

SOLUTION

You can deduce that Austin got a good haircut.

4. Prove that the sum of two even integers is always even.

SOLUTION

Let $2x$ and $2y$ represent any two even numbers.

$$2x + 2y = 2(x+y)$$

Since 2 is a factor of the sum, the sum is therefore even.

5. Prove that the product of an even integer and an odd integer is always even.

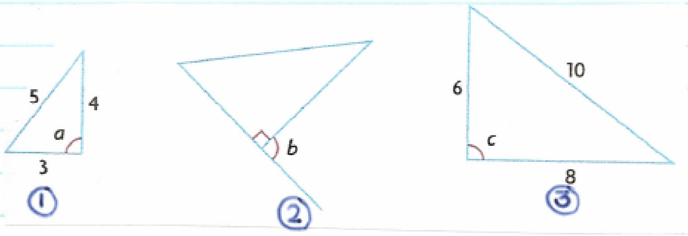
SOLUTION

Let $2x+1$ represent an odd number
Let $2y$ represent an even number

$$2y(2x+1) = 4xy + 2y$$
$$\Rightarrow 2(2xy+y)$$

Since 2 is a factor of the product, the product is even.

6. Prove that a, b, and c are equal.



② We know Angle b and the right angle are supplementary (add to 180°). Therefore, Angle b = 90°.
We can check to see if Angle A and Angle C are also 90° by using the Pythagorean Theorem.

① $c^2 = a^2 + b^2$
$5^2 = 3^2 + 4^2$
$25 = 9 + 16$
$25 = 25$
Therefore, Angle $a = 90°$

③ $c^2 = a^2 + b^2$
$10^2 = 6^2 + 8^2$
$100 = 36 + 64$
$100 = 100$
Therefore, Angle $c = 90°$

* Since Angles a, b, and c are all 90°, they are all equal.

7. Drew created this step-by-step number trick:

- Choose any number
- Multiply by 4
- Add 10
- Divide by 2
- Subtract 5
- Divide by 2
- Add 3

Prove deductively that the result is always 3 more than the chosen number.

SOLUTION

n	n	← Chosen Number
$\times 4$	$4n$	
$+10$	$4n+10$	
$\div 2$	$2n+5$	
-5	$2n$	
$\div 2$	n	
$+3$	$n+3$	← 3 more than the chosen number

10. Prove that whenever you square an odd integer, the result is odd.

SOLUTION

Let $2x+1$ represent any odd integer.

$$
\begin{aligned}
(2x+1)^2 &= (2x+1)(2x+1) \\
&= 4x^2+2x+2x+1 \\
&= \underline{4}x^2+\underline{4}x+1
\end{aligned}
$$

The numbers $4x^2$ and $4x$ are even. The addition of 1 makes the result odd.

16. Look for a pattern when any odd number is squared and then divided by 4.
Make a conjecture, and then prove your conjecture.

Evidence

$\frac{7^2}{4}$ $\frac{17^2}{4}$ $\frac{21^2}{4}$

$= \frac{49}{4}$ $= \frac{289}{4}$ $= \frac{441}{4}$

$= 12.25$ $= 72.25$ $= 110.25$

Conjecture

When an odd number is squared and divided by 4, it will always result in a decimal ending with 0.25.

Proof

Let $2x+1$ represent any odd number

$$\frac{(2x+1)^2}{4} = \frac{(2x+1)(2x+1)}{4}$$
$$= \frac{4x^2+2x+2x+1}{4}$$
$$= \frac{4x^2+4x+1}{4}$$
$$= x^2+x+\frac{1}{4}$$
$$= x^2+x+0.25$$

17. Simon made the following conjecture: When you add three consecutive numbers, your answer is always a multiple of 3. Joan, Garnet, and Jamie took turns presenting their work to prove Simon's conjecture. Which student had the strongest proof? Explain.

Joan's Work	Garnet's Work	Jamie's Work
$1 + 2 + 3 = 6$ $\quad 3 \cdot 2 = 6$ $2 + 3 + 4 = 9$ $\quad 3 \cdot 3 = 9$ $3 + 4 + 5 = 12$ $\quad 3 \cdot 4 = 12$ $4 + 5 + 6 = 15$ $\quad 3 \cdot 5 = 15$ $5 + 6 + 7 = 18$ $\quad 3 \cdot 6 = 18$ and so on ...	$3 + 4 + 5$ The two outside numbers (3 and 5) add to give twice the middle number (4). All three numbers add to give 3 times the middle number.	Let the numbers be n, $n + 1$, and $n + 2$. $n + n + 1 + n + 2 = 3n + 3$ $n + n + 1 + n + 2 = 3(n + 1)$
Simon's conjecture is valid.	Simon's conjecture is valid.	Simon's conjecture is valid.

SOLUTION

Joan and Garnet both used inductive reasoning to provide more evidence for the conjecture, but their solutions are not mathematical proofs.
Jamie had the strongest proof since he used deductive reasoning to develop a generalization that proves Simon's conjecture.

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1s4e2 finalt.mp4

1s4e3 finalt.mp4

1s4e4 finalt.mp4

1s4e5 finalt.mp4