

### Questions from Homework

Exercise 2.8

④ If  $f(x) = \sqrt{1+x^3}$  find  $f''(a)$

$$f(x) = (1+x^3)^{1/2}$$

$$f'(x) = \frac{1}{2}(1+x^3)^{-1/2}(3x^2)$$

$$f'(x) = \frac{3x^2}{2(1+x^3)^{1/2}}$$

$$f''(x) = \frac{6x(2(1+x^3)^{1/2}) - 3x^2(1+x^3)^{-1/2}(3x^2)}{4(1+x^3)}$$

$$f''(x) = \frac{12x(1+x^3)^{1/2} - 9x^4(1+x^3)^{-1/2}}{4(1+x^3)}$$

$$f''(x) = \frac{3x(1+x^3)^{-1/2} [4(1+x^3) - 3x^3]}{4(1+x^3)}$$

$\downarrow$   
 $4 + 4x^3 - 3x^3$

$$f''(x) = \frac{3x(4+x^3)}{4(1+x^3)^{3/2}}$$

$$f''(a) = \frac{3(a)(4+(a)^3)}{4(1+(a)^3)^{3/2}}$$

$$f''(a) = \frac{7a}{108}$$

$f''(a) = \frac{a}{3}$

**Questions from Homework**

⑧ Find a quadratic function  $f$  such that

$$f(3) = 33$$

$$f'(3) = 22$$

$$f''(3) = 8$$

$$f(x) = 4x^2 - 2x + 3$$

$$f'(x) = 8x - 2$$

$$f''(x) = 8$$

## Differentiation Rules

### Product Rule:

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**The Product Rule** If  $f$  and  $g$  are both differentiable, then

$$\frac{d}{dx} [f(x)g(x)] = f(x) \frac{d}{dx} [g(x)] + g(x) \frac{d}{dx} [f(x)]$$

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Express the product rule verbally if you are considering a function of the form...

$$f(x) = (\text{First}) \times (\text{Second})$$

" The derivative of the product of two functions is the the first multiplied by the derivative of second, plus the derivative of first multiplied by the second"

*Get in the habit of verbalizing the rule as you differentiate...it will help when the functions get more complicated.*

## Quotient Rule:

**The Quotient Rule** If  $f$  and  $g$  are differentiable, then

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}$$

Express the quotient rule verbally ...

" The denominator multiplied by the derivative of the numerator, minus the numerator multiplied by the derivative of the denominator, all over the denominator squared"



## Combining the Chain Rule With the Product and Quotient Rule:

**The Chain Rule** If  $f$  and  $g$  are both differentiable and  $F = f \circ g$  is the composite function defined by  $F(x) = f(g(x))$ , then  $F$  is differentiable and  $F'$  is given by the product

$$F'(x) = f'(g(x))g'(x)$$

Differentiate the following function and simplify your answer:

$$y = (x^2 + 1)^3 (2 - 3x)^4$$

$$\begin{aligned} y' &= (x^2+1)^3 (4)(2-3x)^3 (-3) + 3(x^2+1)^2 (2x)(2-3x)^4 \\ &= -12(x^2+1)^3 (2-3x)^3 + 6x(x^2+1)^2 (2-3x)^4 \\ &= -6(x^2+1)^2 (2-3x)^3 \left[ 2(x^2+1) - x(2-3x) \right] \\ &= -6(x^2+1)^2 (2-3x)^3 \left[ 2x^2+2-2x+3x^2 \right] \\ &= \boxed{-6(x^2+1)^2 (2-3x)^3 (5x^2-2x+2)} \end{aligned}$$

$$g(x) = \frac{(3x+2)^2}{2x}$$

$$\begin{aligned} g'(x) &= \frac{2x(2)(3x+2)(3) - (3x+2)^2(2)}{(2x)^2} \\ &= \frac{12x(3x+2) - 2(3x+2)^2}{4x^2} \\ &= \frac{2(3x+2) \left[ 6x - (3x+2) \right]}{4x^2} \\ &= \frac{\cancel{2}(3x+2)(3x-2)}{4x^2} \\ &= \boxed{\frac{(3x+2)(3x-2)}{2x^2}} \quad \text{or} \quad \frac{9x^2-4}{2x^2} \end{aligned}$$

Differentiate the following functions and simplify your answers:

$$s = \left( \frac{2t-1}{t+2} \right)^6$$

$$\frac{ds}{dt} = 6 \left[ \frac{2t-1}{t+2} \right]^5 \left[ \frac{2t+4 - 2t+1}{(t+2)^2} \right]$$

$$\frac{ds}{dt} = 6 \left[ \frac{(2t-1)^5}{(t+2)^5} \right] \left[ \frac{5}{(t+2)^2} \right]$$

$$\frac{ds}{dt} = \frac{30(2t-1)^5}{(t+2)^7}$$

$$g(x) = (9x^{-3})(5x^3 - 1)^6$$

$$g'(x) = (9x^{-3})[6(5x^3-1)^5(15x^2)] - 27x^{-4}(5x^3-1)^6$$

$$g'(x) = 810x^{-1}(5x^3-1)^5 - 27x^{-4}(5x^3-1)^6$$

$$g'(x) = 27x^{-4}(5x^3-1)^5 \left[ 30x^3 - 5x^3 + 1 \right]$$

$$g'(x) = 27x^{-4}(5x^3-1)^5(25x^3+1)$$

$$g'(x) = \frac{27(5x^3-1)^5(25x^3+1)}{x^4}$$

$$\begin{array}{lll} F(x) = f(x)g(x) & \text{product} & F(x) = (f \cdot g)(x) \quad f'(x)g(x) + f(x)g'(x) \\ F(x) = \frac{f(x)}{g(x)} & \text{quotient} & F(x) = \left(\frac{f}{g}\right)(x) \quad \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \\ F(x) = f(g(x)) & \text{chain} & F(x) = (f \circ g)(x) \quad f'(g(x))g'(x) \end{array}$$

**Example 1**Chain Rule  
↙Let  $F(x) = f(g(x))$ If  $f(2) = 3$ ,  $f'(2) = 5$ ,  $g(1) = 2$  and  $g'(1) = 4$  find  $F'(1)$ .

$$F'(x) = f'(g(x))g'(x)$$

$$F'(1) = f'(\underline{g(1)})\underline{g'(1)}$$

$$F'(1) = \underline{f'(2)} \cdot (4)$$

$$F'(1) = (5) \cdot (4)$$

$$F'(1) = 20$$

**Example 2**

If  $y = u^{10} + u^5 + 2$ , where  $u = 1 - 3x^2$ , find  $\left. \frac{dy}{dx} \right|_{x=1}$

①  $\frac{dy}{du} = 10u^9 + 5u^4$       ②  $\frac{du}{dx} = -6x$       ③ when  $x = 1$   
 $u = 1 - 3x^2$   
 $u = 1 - 3(1)^2$   
 $u = -2$

④  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\left. \frac{dy}{dx} \right|_{x=1} = [10u^9 + 5u^4] [-6x]$$

$$\left. \frac{dy}{dx} \right|_{x=1} = [10(-2)^9 + 5(-2)^4] [-6(1)]$$

$$\left. \frac{dy}{dx} \right|_{x=1} = [-5120 + 80] [-6]$$

$$\left. \frac{dy}{dx} \right|_{x=1} = [-5040] [-6]$$

$$\left. \frac{dy}{dx} \right|_{x=1} = 30240$$

# Homework

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$$\textcircled{4} \text{ e) } y = \sqrt{x}(5 - \sqrt{x}) = 5\sqrt{x} - x = 5x^{1/2} - x$$

$$y' = \frac{5}{2}x^{-1/2} - 1$$

$$y' = \frac{5}{2\sqrt{x}} - 1 = \frac{5 - 2\sqrt{x}}{2\sqrt{x}}$$

Using Product:

$$y = \sqrt{x}(5 - \sqrt{x})$$

$$y' = \frac{1}{2}x^{-1/2}(5 - \sqrt{x}) + x^{1/2}\left(-\frac{1}{2}x^{-1/2}\right)$$

$$y' = \frac{1}{2\sqrt{x}}(5 - \sqrt{x}) + \sqrt{x}\left(-\frac{1}{2\sqrt{x}}\right)$$

$$y' = \frac{5}{2\sqrt{x}} - \frac{\sqrt{x}}{2\sqrt{x}} - \frac{\sqrt{x}}{2\sqrt{x}}$$

$$y' = \frac{5}{2\sqrt{x}} - \frac{2\sqrt{x}}{2\sqrt{x}}$$

$$y' = \frac{5}{2\sqrt{x}} - 1$$

$$\textcircled{5} \text{ f) } y = \frac{x^2 - 2x}{\sqrt{x}}$$

$$y' = \frac{x^{1/2}(2x - 2) - \frac{1}{2}x^{-1/2}(x^2 - 2x)}{x \cdot 2x^{1/2}}$$

$$y' = \frac{2x^{3/2} - 2x^{1/2} - \frac{1}{2}(x^2 - 2x) \cdot \frac{1}{2x^{1/2}}}{x \cdot 2x^{1/2}}$$

$$y' = \frac{4x^2 - 4x - x^2 + 2x}{2x^{3/2}} = \frac{3x^2 - 2x}{2x^{3/2}}$$

$$= \frac{x(3x - 2)}{2x^{3/2}}$$

$$= \frac{x^{-1/2}(3x - 2)}{2}$$

$$= \boxed{\frac{3x - 2}{2\sqrt{x}}}$$

$$\textcircled{4} \text{ r) } f(y) = \frac{1}{2 + \frac{3}{y}}$$

$$f(y) = \left(2 + \frac{3}{y}\right)^{-1}$$

$$f'(y) = -\left(2 + \frac{3}{y}\right)^{-2} (-3y^{-2})$$

$$f'(y) = \frac{3}{y^2 \left(2 + \frac{3}{y}\right)^2}$$


$$f'(y) = \frac{3}{y^2 \left(4 + \frac{12}{y} + \frac{9}{y^2}\right)}$$

$$f'(y) = \frac{3}{4y^2 + 12y + 9}$$

$$\underline{6} \times \underline{6} = 36$$

$$\underline{6} + \underline{6} = 12$$

$$f'(y) = \frac{3}{\left(y + \frac{6}{4}\right)\left(y + \frac{6}{4}\right)}$$

$$f'(y) = \frac{3}{\left(y + \frac{3}{2}\right)\left(y + \frac{3}{2}\right)}$$

$$f'(y) = \frac{3}{(2y+3)(2y+3)}$$

$$f'(y) = \frac{3}{(2y+3)^2}$$



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⑥ if  $y = u^2 - u^3 + 2u^4$  and  $u = \frac{x}{2x-1}$  Find  $\frac{dy}{dx} \Big|_{x=1}$

(i)  $\frac{dy}{du} = \underline{2u - 3u^2 + 8u^3}$

(ii)  $\frac{du}{dx} = \frac{1(2x-1) - 2x}{(2x-1)^2}$

(iii) Find  $u$

$$u = \frac{x}{2x-1}$$

$$u = \frac{(1)}{2(1)-1}$$

(iv)  $\frac{dy}{dx} \Big|_{x=1} = \left[ \frac{dy}{du} \right] \cdot \left[ \frac{du}{dx} \right]$

$$\frac{dy}{dx} \Big|_{x=1} = \left[ 2u - 3u^2 + 8u^3 \right] \left[ \frac{-1}{(2x-1)^2} \right]$$

$$\frac{dy}{dx} \Big|_{x=1} = \left[ 2(1) - 3(1)^2 + 8(1)^3 \right] \left[ \frac{-1}{(2(1)-1)^2} \right]$$

$$\frac{dy}{dx} \Big|_{x=1} = \left[ 2 - 3 + 8 \right] \left[ -1 \right] = \underline{-7}$$

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$$\text{Ex: } y = 7x^2$$

$$\frac{dy}{dx} = 14x$$

① b)  $x^2 - (x^2 y) + y^2 = 1$

*Product*

$$\frac{d}{dx} (x^2 - (x^2 y) + y^2) = 0$$

$$\frac{d}{dx} x^2 - \frac{d}{dx} (x^2 y) + \frac{d}{dx} y^2 = 0$$

$$2x - x^2 \frac{dy}{dx} + 2y \frac{dy}{dx} = 2xy - 2x$$

$$\frac{dy}{dx} (2y - x^2) = \frac{2xy - 2x}{2y - x^2}$$

$$\frac{dy}{dx} = \frac{2xy - 2x}{2y - x^2}$$

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$$\textcircled{1} \text{ d) } y\sqrt{x-1} + x\sqrt{y-1} = xy$$

$$y(x-1)^{\frac{1}{2}} + x(y-1)^{\frac{1}{2}} = xy$$

$$\frac{dy}{dx}(x-1)^{\frac{1}{2}} + y\left(\frac{1}{2}\right)(x-1)^{-\frac{1}{2}}(1) + 1(y-1)^{\frac{1}{2}} + x\left(\frac{1}{2}\right)(y-1)^{-\frac{1}{2}}\left(\frac{dy}{dx}\right) = 1y + x\frac{dy}{dx}$$

$$\frac{dy}{dx}(x-1)^{\frac{1}{2}} + \frac{x}{2}(y-1)^{-\frac{1}{2}}\frac{dy}{dx} - x\frac{dy}{dx} = y - \frac{y}{2}(x-1)^{-\frac{1}{2}} - (y-1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} \left[ (x-1)^{\frac{1}{2}} + \frac{x}{2}(y-1)^{-\frac{1}{2}} - x \right] = y - \frac{y}{2}(x-1)^{-\frac{1}{2}} - (y-1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{y - \frac{y}{2}(x-1)^{-\frac{1}{2}} - (y-1)^{\frac{1}{2}}}{(x-1)^{\frac{1}{2}} + \frac{x}{2}(y-1)^{-\frac{1}{2}} - x}$$

$$\frac{dy}{dx} = \frac{y - \frac{y}{2\sqrt{x-1}} - \sqrt{y-1}}{\sqrt{x-1} + \frac{x}{2\sqrt{y-1}} - x}$$

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$$\textcircled{8} \text{ d) } x^2 + y^2 = \underline{\underline{16}}$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$\frac{2y \frac{dy}{dx}}{2y} = -\frac{\cancel{2x}}{\cancel{2y}}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = \frac{-1(y) + x \left( \frac{dy}{dx} \right)}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-y + x \left( -\frac{x}{y} \right)}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-y^2 - \frac{x^2}{y}}{y^2 y}$$

$$\frac{d^2y}{dx^2} = \frac{-y^2 - x^2}{y^3}$$

$$\frac{d^2y}{dx^2} = -\frac{y^2 + x^2}{y^3}$$

$$\frac{d^2y}{dx^2} = -\frac{16}{y^3}$$

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9) c)  $y = \frac{1}{\sqrt{x^5}}$ ,  $(\underline{2}, \underline{\frac{1}{4\sqrt{2}}})$

c) Find derivative:

$$y = \frac{1}{\sqrt{x^5}}$$

$$y' = \frac{-5}{2\sqrt{x^7}}$$

$$y = \frac{1}{x^{5/2}}$$

$$y' = \frac{-5}{2\sqrt{(2)^7}}$$

$$y = x^{-5/2}$$

$$y' = \frac{-5}{2\sqrt{128}}$$

$$y' = \frac{-5}{2} x^{-7/2}$$

$$y' = \frac{-5}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$$

$$y' = \frac{-5}{2\sqrt{x^7}}$$

$$y' = \frac{-5}{16\sqrt{2}}$$

(ii) Find equation

$$y - y_1 = m(x - x_1)$$

$$y - \frac{1}{4\sqrt{2}} = \frac{-5}{16\sqrt{2}}(x - 2)$$

$$y - \frac{1}{4\sqrt{2}} = \frac{-5x + 10}{16\sqrt{2}}$$

$$y = \frac{-5x + 10}{16\sqrt{2}} + \frac{1}{4\sqrt{2}}$$

$$y = \frac{-5x}{16\sqrt{2}} + \frac{10}{16\sqrt{2}} + \frac{4}{16\sqrt{2}}$$

multiply by  $16\sqrt{2}$

$$16y\sqrt{2} = -5x + 14$$

$$5x + 16y\sqrt{2} - 14 = 0$$

or

$$y = \frac{-5x}{16\sqrt{2}} + \frac{7}{8\sqrt{2}}$$

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$$\textcircled{1} \text{ d), } y = x\sqrt{x^2+5}, \quad (-2, -6) \quad \begin{array}{l} x_1 = -2 \\ y_1 = -6 \end{array}$$

① Find  $y'$ 

$$y' = \sqrt{x^2+5} + x \left[ \frac{1}{2} (x^2+5)^{-\frac{1}{2}} \cdot 2x \right]$$

$$y' = \sqrt{x^2+5} + \frac{x^2}{\sqrt{x^2+5}}$$

② Find  $m$  or  $y'(-2)$ 

$$y'(-2) = \sqrt{(-2)^2+5} + \frac{(-2)^2}{\sqrt{(-2)^2+5}}$$

$$y' = 3 + \frac{4}{3} = \boxed{\frac{13}{3}}$$

③ Find equation:

$$y + 6 = \frac{13}{3}(x + 2)$$

$$y + 6 = \frac{13x}{3} + \frac{26}{3} - 6$$

$$\boxed{y = \frac{13x}{3} + \frac{8}{3}}$$

$$3y = 13x + 8$$

$$\boxed{0 = 13x - 3y + 8}$$

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12) Find the points on the curve  $y = \frac{1}{2x-1}$  where the tangent line is perpendicular to  $x-2y=1$

(i) Find perpendicular slope:

$$x-2y=1$$

$$-2y = -x + 1$$

$$y = \frac{-x}{-2} + \frac{1}{-2}$$

$$y = \left(\frac{1}{2}\right)x - \frac{1}{2}$$

$$m = \frac{1}{2}$$

$$m_{\perp} = -2$$

(ii) Find derivative:

$$y = \frac{1}{2x-1} = (2x-1)^{-1}$$

$$y' = -1(2x-1)^{-2}(2)$$

$$y' = \frac{-2}{(2x-1)^2}$$

(iii) Solve for x

$$\frac{-2}{(2x-1)^2} = -2$$

$$-2(2x-1)^2 = -2$$

$$(2x-1)^2 = 1$$

$$4x^2 - 4x + 1 = 1$$

$$4x^2 - 4x = 0$$

$$4x(x-1) = 0$$

$$4x = 0 \quad | \quad x-1 = 0$$

$$x = 0 \quad | \quad x = 1$$

(iv) Solve for y:

$$\text{if } x=0$$

$$y = \frac{1}{2x-1}$$

$$y = \frac{1}{-1} = -1$$

$$(0, -1)$$

$$\text{if } x=1$$

$$y = \frac{1}{2x-1}$$

$$y = \frac{1}{1} = 1$$

$$(1, 1)$$

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⑭ Suppose a) find  $(fg)'(3)$  Product

$$f(3) = \underline{4}$$

$$f'(3) = \underline{-1}$$

$$f'(6) = \underline{5}$$

$$g(3) = \underline{6}$$

$$g'(3) = \underline{2}$$

$$(fg)'(3) = \underline{f'(3)} \underline{g(3)} + \underline{f(3)} \underline{g'(3)}$$

$$= (-1)(6) + (4)(2)$$

$$= -6 + 8$$

$$= 2$$

c) find  $(f \circ g)'(3)$

$$(f \circ g)'(3) = \underline{f'(g(3))} \cdot \underline{g'(3)}$$

$$= \underline{f'(6)} \cdot (2)$$

$$= (5) \cdot (2)$$

$$= 10$$



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③ Find  $\frac{dy}{dx}$   $\Big|_{x=4}$  if  $y = u^2 - 2u^5$  and  $u = x - \sqrt{x}$

① Find  $u$ 

$$u = x - \sqrt{x}$$

$$u = (4) - \sqrt{4}$$

$$u = 2$$

② Find  $\frac{dy}{du}$ 

$$y = u^2 - 2u^5$$

$$\frac{dy}{du} = 2u - 10u^4$$

③ Find  $\frac{du}{dx}$ 

$$u = x - x^{1/2}$$

$$\frac{du}{dx} = 1 - \frac{1}{2}x^{-1/2}$$

$$\begin{aligned} \frac{dy}{dx} \Big|_{x=4} &= (2u - 10u^4) \left(1 - \frac{1}{2\sqrt{x}}\right) \\ &= (2(2) - 10(2)^4) \left(1 - \frac{1}{2\sqrt{4}}\right) \\ &= (4 - 160) \left(1 - \frac{1}{4}\right) \\ &= (-156) \left(\frac{3}{4}\right) \\ &= \frac{-468}{4} \\ &= -117 \end{aligned}$$

⑨ If  $F(x) = f(g(x))$ , where  $\begin{matrix} g(a) = 4 \\ g'(a) = 3 \\ f'(4) = 5 \end{matrix}$  find  $F'(a)$

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$F'(a) = f'(g(a)) \cdot g'(a)$$

$$F'(a) = f'(4) \cdot g'(a)$$

$$F'(a) = 5 \cdot 3 = \underline{\underline{15}}$$

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# 3-10

$$\textcircled{4} \left. \frac{dy}{dt} \right]_{t=1}$$

$$y = \sqrt{1+r^2}$$

$$\frac{dy}{dr} = \frac{1}{2}(1+r^2)^{-1/2} (2r)$$

$$\frac{dy}{dr} = \frac{r}{\sqrt{1+r^2}}$$

$$\frac{dr}{dt} = \frac{1(2t+1) - 2(t+1)}{(2t+1)^2}$$

$$\frac{dr}{dt} = \frac{-1}{(2t+1)^2}$$

$$r = \frac{t+1}{2t+1}$$

$$\left. \frac{dy}{dt} \right]_{t=1} = \left[ \frac{dy}{dr} \right] \left[ \frac{dr}{dt} \right]$$

$$= \left[ \frac{r}{\sqrt{1+r^2}} \right] \left[ \frac{-1}{(2t+1)^2} \right]$$

$$= \left[ \frac{2/3}{\sqrt{1+(2/3)^2}} \right] \left[ \frac{-1}{(2(1)+1)^2} \right]$$

$$= \left[ \frac{2/3}{\sqrt{13/9}} \right] \left[ -\frac{1}{9} \right]$$

$$= \left[ \frac{2}{3} \cdot \frac{3}{\sqrt{13}} \right] \left[ -\frac{1}{9} \right]$$

$$= \boxed{\frac{-2}{9\sqrt{13}}}$$

when  $t=1$   $r = \frac{2}{3}$

$$\text{If } f(3) = -2, f'(3) = 3, g(3) = 1, g'(3) = 7 \\ \text{and } f'(1) = 4$$

Find:

$$\begin{aligned} \text{(i) } (f \circ g)(3) &= f'(3)g(3) + f(3) \cdot g'(3) \\ &= (3)(1) + (-2)(7) \\ &= 3 - 14 \\ &= -11 \end{aligned}$$

$$\begin{aligned} \text{(ii) } \left(\frac{f}{g}\right)'(3) &= \frac{f'(3)g(3) - g'(3)f(3)}{[g(3)]^2} \\ &= \frac{(3)(1) - (7)(-2)}{(1)^2} \\ &= \frac{3 + 14}{1} \\ &= 17 \end{aligned}$$

$$\begin{aligned} \text{(iii) } (f \circ g)'(3) &= f'(g(3)) \cdot g'(3) \\ &= f'(1) \cdot g'(3) \\ &= 4 \cdot 7 \\ &= 28 \end{aligned}$$