

Warm Up

Differentiate the following:

$$f(x) = \frac{(-2x)(\tan^{-1} \sqrt{x})}{\cos^{-1}(\sec x^3)}$$

$$f'(x) = \underbrace{\left[-2 \tan^{-1} \sqrt{x} - 2x \left(\frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} \right) \right]}_{f'(x)} \underbrace{\left[\cos^{-1}(\sec x^3) \right]}_{g(x)} +$$

$$\underbrace{\left[2x \tan^{-1} \sqrt{x} \right]}_{f(x)} \underbrace{\left[\frac{-1}{\sqrt{1-\sec^2 x^3}} \cdot \sec x^3 \tan x^3 \cdot 3x^2 \right]}_{g'(x)}$$

$$\underbrace{\left[\cos^{-1}(\sec x^3) \right]^2}_{g(x)}$$

Questions from Homework

$$\textcircled{1} \text{ e) } y = \cos^{-1}\left(\frac{x^3}{2}\right) = \cos^{-1}\left(\frac{1}{2}x^3\right)$$

$$y' = \frac{-1}{\sqrt{1 - \left(\frac{x^3}{2}\right)^2}} \cdot \frac{3x^2}{2}$$

$$y' = \frac{-3x^2}{2\sqrt{1 - \frac{x^6}{4}}}$$

$$y' = \frac{-3x^2}{\sqrt{4\left(1 - \frac{x^6}{4}\right)}} = \frac{-3x^2}{\sqrt{4 - x^6}}$$

Questions from Homework

$$\textcircled{1} \text{ n) } y = \frac{\sqrt{1-x^2}}{x} + \sin^{-1} x$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}(1-x^2)^{-1/2}(-2x)(x) - (1-x^2)^{-1/2}(1)}{x^2} + \frac{1}{\sqrt{1-x^2}} \cdot 1$$

$$\frac{dy}{dx} = \frac{-x^2(1-x^2)^{-1/2} - (1-x^2)^{-1/2}}{x^2} + \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-(1-x^2)^{-1/2} (x^2 + 1 - x^2)}{x^2} + \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-1}{x^2 \sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-1}{x^2 \sqrt{1-x^2}} + \frac{x^2}{x^2 \sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{x^2 - 1}{x^2 \sqrt{1-x^2}}$$

Questions from Homework

$$\textcircled{1} p) y = \sin^{-1}(x) + \cos^{-1} \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \cdot 1 - \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{1}{2} (1-x^2)^{-1/2} (-2x)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-(1-x^2)} \sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{x^2} \sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{|x| + x}{|x| \sqrt{1-x^2}}$$

Questions from Homework

$$\textcircled{2} f(x) = x \tan^{-1} x$$

where $x=1$

(i) Differentiate:

$$f(x) = x \tan^{-1} x$$

$$f'(x) = 1 \tan^{-1} x + x \left(\frac{1}{1+x^2} \cdot 1 \right)$$

$$f'(x) = \tan^{-1} x + \frac{x}{1+x^2}$$

(ii) Solve $f'(1)$ what angle has a tan value of 1

$$f'(1) = \boxed{\tan^{-1}(1)} + \frac{(1)}{1+(1)^2}$$

(radians)

$$f'(1) = \frac{\pi}{4} + \frac{1}{2}$$

$$\boxed{f'(1) = \frac{\pi + 2}{4}}$$

$$\textcircled{6} \text{ IF } f(x) = (x-3)\sqrt{6x-x^2} + 9 \sin^{-1} \left(\frac{x-3}{3} \right) \text{ Find } f'(3)$$

$\frac{x-3}{3} \rightarrow \frac{x}{3} - \frac{3}{3}$
 $\frac{1}{3}x - 1$

$$f'(x) = 1\sqrt{6x-x^2} + (x-3) \frac{1}{2} (6x-x^2)^{-1/2} (6-2x) + 9 \left(\frac{1}{\sqrt{1-(\frac{x-3}{3})^2}} \cdot \frac{1}{3} \right)$$

$$f'(x) = \sqrt{6x-x^2} + \frac{(x-3)(3-x)}{\sqrt{6x-x^2}} + \frac{3}{\sqrt{1-\frac{(x-3)^2}{9}}}$$

$$f'(3) = \sqrt{18-9} + \frac{(0)(0)}{\sqrt{18-9}} + \frac{3}{\sqrt{1-\frac{0}{9}}}$$

$$f'(3) = 3 + 0 + 3 = \boxed{6}$$

Questions from Homework

$$\textcircled{4} f(x) = (3 \tan^{-1} x)^4$$

$$f'(x) = 4(3 \tan^{-1} x)^3 \left[3 \left(\frac{1}{1+x^2} \cdot 1 \right) + \cancel{(0) \tan^{-1} x} \right]$$

$$f'(x) = 4(3 \tan^{-1} x)^3 \left[\frac{3}{1+x^2} \right]$$

$$f'(x) = \frac{12(3 \tan^{-1} x)^3}{1+x^2}$$

$$f'(\sqrt{3}) = \frac{12(3 \tan^{-1} \sqrt{3})^3}{1+(\sqrt{3})^2}$$

what angle has a tangent value equal to $\sqrt{3}$

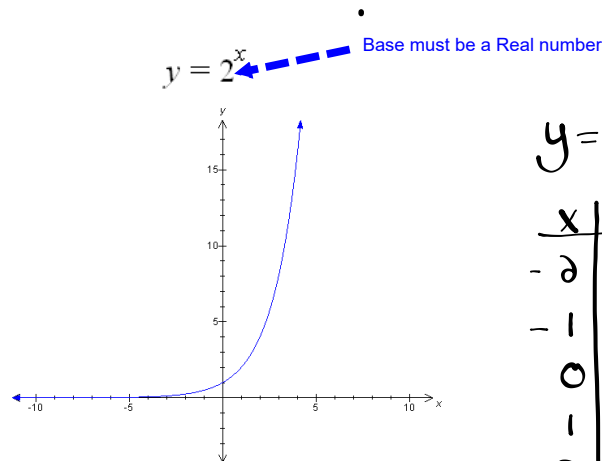
$$= \frac{12(3(\frac{\pi}{3}))^3}{1+3}$$

$$= \frac{12\pi^3}{4}$$

$$= 3\pi^3$$

Differentiating Exponential Functions

What is an exponential function?



$y = 2^x$

x	y
-2	1/4
-1	1/2
0	1
1	2
2	4

When you do not have a rule to differentiate resort to the definition...

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

(1) $f(x+h) = a^{x+h}$

Let's try and differentiate $y = a^x$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h}$$

This factor does not depend on h , therefore we can move to the front of the limit

$$= \lim_{h \rightarrow 0} \frac{a^x a^h - a^x}{h} = \lim_{h \rightarrow 0} \frac{a^x(a^h - 1)}{h}$$

Thus we now have...

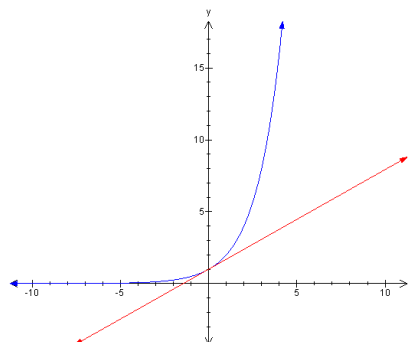
$$f'(x) = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$$

What would be the value of $f'(0)$?

$a^0 = 1$

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = f'(0)$$

What would this represent in terms of slope??



Draw your tangent to the curve at $x=0$

We have determined that $f'(x) = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$

and that $\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = f'(0)$

Same thing!

Therefore given $f(x) = a^x$, then $f'(x) = a^x f'(0)$

Here are a couple of numerical examples...

- $a = 2$; here apparently $f'(0) \approx 0.69$
- $a = 3$; here apparently $f'(0) \approx 1.10$

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

There must then be some number between 2 and 3 such that

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = 1$$

This number turns out to be "e"...Euler's Number

```
e^(1)
2.718281828
```

$$\frac{\log 243}{\log 3} = 5 \quad \left| \quad \frac{\ln 243}{\ln 3} = 5$$

$$3^5 = 243 \quad \left| \quad 3^5 = 243$$

This leads to the following definition...

Definition of the Number e

$$e \text{ is the number such that } \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

What does this mean geometrically?

- Geometrically, this means that
 - of all the exponential functions $y = a^x$,
 - the function $f(x) = e^x$ is the one whose tangent at $(0, 1)$ has a slope $f'(0)$ that is exactly 1.

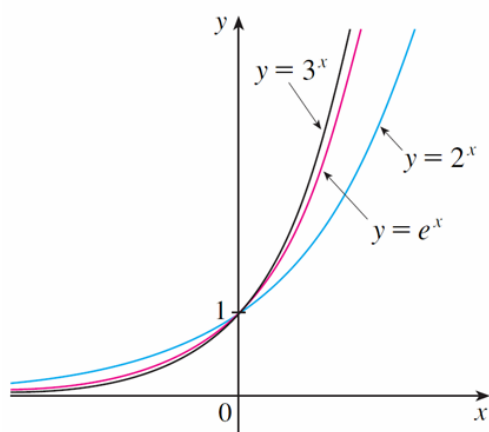


FIGURE 6

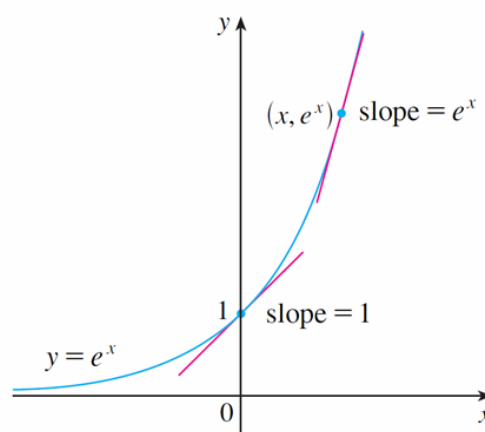


FIGURE 7

This leads to the following differentiation formula...

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

**This is the ONLY function
that is its own derivative**

$$f(x) = e^x$$
$$f'(x) = e^x$$

In General...

$$\frac{d(e^u)}{dx} = e^u \bullet du$$

Differentiating Exponential Functions

$$y = e^{3x^7} \quad u = 3x^7 \quad du = 21x^6$$

$$y' = e^{3x^7} \cdot 21x^6$$

$$y' = 21x^6 e^{3x^7}$$

$$y = e^{\sin x} \quad u = \sin x \quad du = \cos x$$

$$y' = e^{\sin x} \cdot \cos x$$

$$y' = \cos x e^{\sin x}$$

$$y = (x^2)(e^x)$$

$$y' = 2x(e^x) + x^2 e^x$$

$$y' = 2xe^x + x^2 e^x$$

$$y' = xe^x(2+x)$$

$$y = e^{\cot x^3}$$

$$y' = e^{\cot x^3} \cdot (-\csc^2(x^3) \cdot 3x^2)$$

$$y' = -3x^2 \csc^2(x^3) e^{\cot x^3}$$

Practice Exercises

Page 367

#4, 5, 6, 8, 9, 10,

Bonus:

Give that $y = \cos^{-1}(\cos^{-1} x)$, prove that

$$\frac{dy}{dx} = \frac{1}{\sin y \sqrt{1-x^2}}$$