Warm Up

Differentiate the following:

$$f(x)=\frac{(-2x)\left(\tan^{-1}\sqrt{x}\right)}{\cos^{-1}\left(\sec x^{3}\right)}$$

$$f'(x)=\frac{\overbrace{\left[-2\tan^{-1}\sqrt{x}-2x\left(\frac{1}{1+(\sqrt{x})^{2}}\cdot\frac{1}{2\sqrt{x}}\right)\right]}^{f'(x)}\overbrace{\left[\cos^{-1}(\sec x^{3})\right]}^{g(x)}+\underbrace{\left[2x\tan^{-1}\sqrt{x}\right]}_{f(x)}\underbrace{\left[\frac{-1}{\sqrt{1-\sec^{2}x^{3}}}\cdot\sec x^{3}\tan x^{3}\cdot 3x^{2}\right]}_{g'(x)}}{\underbrace{\left[\cos^{-1}(\sec x^{3})\right]^{2}}_{g(x)}}$$

Questions from Homework

① e) $y = \cos^{-1}\left(\frac{x^3}{2}\right) = \cos^{-1}\left(\frac{1}{2}x^3\right)$

$y' = \frac{-1}{\sqrt{1-\left(\frac{x^3}{2}\right)^2}} \cdot \frac{3}{2}x^2$

$y' = \frac{-3x^2}{2\sqrt{1-\frac{x^6}{4}}}$

$y' = \frac{-3x^2}{\sqrt{\underline{4}\left(1-\frac{x^6}{4}\right)}} = \frac{-3x^2}{\sqrt{4-x^6}}$

Questions from Homework

1 n) $y = \dfrac{\sqrt{1-x^2}}{x} + \sin^{-1} x$

$$\frac{dy}{dx} = \frac{\cancel{\frac{1}{2}}(1-x^2)^{-\frac{1}{2}}(-\cancel{2}x)(x) - (1-x^2)^{\frac{1}{2}}(1)}{x^2} + \frac{1}{\sqrt{1-x^2}} \cdot 1$$

$$\frac{dy}{dx} = \frac{-x^2(1-x^2)^{-\frac{1}{2}} - (1-x^2)^{\frac{1}{2}}}{x^2} + \frac{1}{\sqrt{1-x^2}}$$

$x^2 + 1 - x^2$

$$\frac{dy}{dx} = \frac{-(1-x^2)^{-\frac{1}{2}}\left(x^2 + (1-x^2)\right)}{x^2} + \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-1}{x^2\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-1}{x^2\sqrt{1-x^2}} + \frac{x^2}{x^2\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{x^2-1}{x^2\sqrt{1-x^2}}$$

Questions from Homework

① p) $y = \sin^{-1}(x) + \cos^{-1}\sqrt{1-x^2}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}\cdot 1 - \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}}\cdot \frac{1}{\cancel{2}}(1-x^2)^{-1/2}(-\cancel{2}x)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{\underbrace{1-(1-x^2)}_{1-1+x^2}}\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{x^2}\sqrt{1-x^2}}$$

by definition $\sqrt{x^2} = |x|$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{x}{|x|\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{|x|}{|x|\sqrt{1-x^2}} + \frac{x}{|x|\sqrt{1-x^2}}$$

$$\boxed{\frac{dy}{dx} = \frac{|x|+x}{|x|\sqrt{1-x^2}}}$$

if $x>0$ then $\frac{x}{|x|} > 0$

if $x<0$ the $\frac{x}{|x|} < 0$

Questions from Homework

② $f(x) = x\tan^{-1}x$ where $x = 1$

(i) Differentiate:

$f(x) = x\tan^{-1}x$

$f'(x) = 1\tan^{-1}x + x\left(\frac{1}{1+x^2}\cdot 1\right)$

$f'(x) = \tan^{-1}x + \frac{x}{1+x^2}$

(ii) Solve $f'(1)$ what angle has a tan value of 1

$f'(1) = \boxed{\tan^{-1}(1)} + \frac{(1)}{1+(1)^2}$

(radians)

$f'(1) = \frac{\pi}{4} + \frac{1}{2}$

$$\boxed{f'(1) = \frac{\pi + 2}{4}}$$

⑥ If $f(x) = (x-3)\sqrt{6x-x^2} + 9\sin^{-1}\boxed{\left(\frac{x-3}{3}\right)}$ Find $f'(3)$

$\frac{x}{3} - \frac{3}{3}$

$\frac{1}{3}x - 1$

$f'(x) = 1\sqrt{6x-x^2} + (x-3)\cancel{\frac{1}{2}}(6x-x^2)^{-1/2}\cancel{(6-2x)}^{3-x} + \cancel{9}^{3}\left(\frac{1}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}}\cdot\frac{1}{\cancel{3}}\right)$

$f'(x) = \sqrt{6x-x^2} + \frac{(x-3)(3-x)}{\sqrt{6x-x^2}} + \frac{3}{\sqrt{1-\frac{(x-3)^2}{9}}}$

$f'(3) = \sqrt{18-9} + \frac{(0)(0)}{\sqrt{18-9}} + \frac{3}{\sqrt{1-\frac{0}{9}}}$

$f'(3) = 3 + 0 + 3 = \boxed{6}$

$\Rightarrow y' = \frac{1 - y^2\cos x - x^2 y^2 \cos x}{(1+x^2)(1+2y\sin x)}$

6. $f'(x) = \frac{(x-3)(6-2x)}{2\sqrt{6x-x^2}} + \sqrt{6x-x^2} + 9\frac{1}{\sqrt{1-\frac{(x-3)^2}{9}}}\cdot\frac{1}{3}$

$= \frac{(x-3)(3-x)}{\sqrt{6x-x^2}} + \sqrt{6x-x^2} + \frac{9}{\sqrt{-x^2+6x}}.$

$f'(3) = 0 + \sqrt{18-9} + \frac{\cancel{27}\ 9}{\sqrt{-9+18}} = 3 + \cancel{9}\ 3 = \cancel{12}\ 6.$

$u = \frac{x-3}{3}$

$u = \frac{1}{3}x - 1$

$du = \frac{1}{3}$

$u = \frac{x-3}{3}$

$du = \frac{1(3) - \cancel{(x-3)(0)}}{9}$

$du = \frac{3}{9}$

$du = \frac{1}{3}$

$\frac{9}{3\sqrt{1-\frac{(x-3)^2}{9}}}$

$\frac{9}{\sqrt{9\left(1-\frac{(x-3)^2}{9}\right)}}$

$\frac{9}{\sqrt{9-(x-3)^2}}$

$\frac{9}{\sqrt{9-(x^2-6x+9)}}$

$\frac{9}{\sqrt{9-x^2+6x-9}}$

$\frac{9}{\sqrt{-x^2+6x}}$

Questions from Homework

4. $f(x) = (3\tan^{-1}x)^4$

$$f'(x) = 4(3\tan^{-1}x)^3\left[3\left(\frac{1}{1+x^2}\cdot 1\right) + \cancel{(0)\tan^{-1}x}\right]$$

$$f'(x) = 4(3\tan^{-1}x)^3\left[\frac{3}{1+x^2}\right]$$

$$f'(x) = \frac{12(3\tan^{-1}x)^3}{1+x^2}$$

$$f'(\sqrt{3}) = \frac{12(3\tan^{-1}\sqrt{3})^3}{1+(\sqrt{3})^2}$$

what angle has a tangent value equal to $\sqrt{3}$

$$= \frac{12(3(\pi/3))^3}{1+3}$$

$$= \frac{12\pi^3}{4}$$

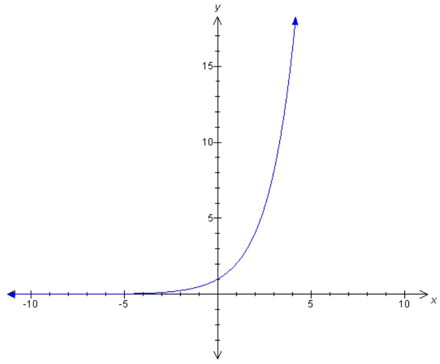
$$\boxed{= 3\pi^3}$$

Differentiating Exponential Functions

What is an exponential function?

•

$y = 2^x$ Base must be a Real number



$y = 2^x$

x	y
-2	1/4
-1	1/2
0	1
1	2
2	4

When you do not have a rule to differentiate resort to the definition...

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(1) $f(x+h) = a^{x+h}$

Let's try and differentiate $y = a^x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h}$$

This factor does not depend on h, therefore we can move to the front of the limit

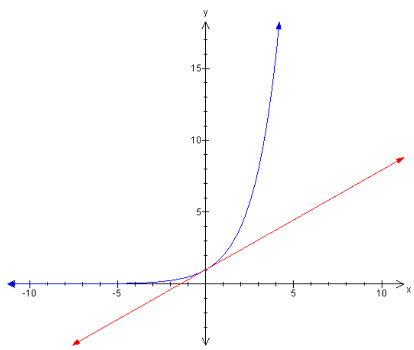
Thus we now have...

$$f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

What would be the value of $f(0)$? $a^0 = 1$

$$\lim_{h \to 0} \frac{a^h - 1}{h} = f'(0)$$

What would this represent in terms of slope??



Draw your tangent to the curve at $x = 0$

We have determined that $f'(x) = a^x \lim_{h\to 0} \frac{a^h - 1}{h}$

Same thing!

and that $\lim_{h\to 0} \frac{a^h - 1}{h} = f'(0)$

Therefore given $f(x) = a^x$, then $f'(x) = a^x f'(0)$

Here are a couple of numerical examples...

- $a = 2$; here apparently $f'(0) \approx 0.69$
- $a = 3$; here apparently $f'(0) \approx 1.10$

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

There must then be some number between 2 and 3 such that

$$\lim_{h\to 0} \frac{a^h - 1}{h} = 1$$

This number turns out to be "*e*"...Euler's Number

```
e^(1)
          2.718281828
```

$\frac{\log 243}{\log 3} = 5$

$3^5 = 243$

$\frac{\ln 243}{\ln 3} = 5$

$3^5 = 243$

This leads to the following definition...

Definition of the Number e

e is the number such that $$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

What does this mean geometrically?

- Geometrically, this means that
  - of all the exponential functions $y = a^x$,
  - the function $f(x) = e^x$ is the one whose tangent at $(0, 1)$ has a slope $f'(0)$ that is exactly 1 .

$y = 1$

$y' = 1$

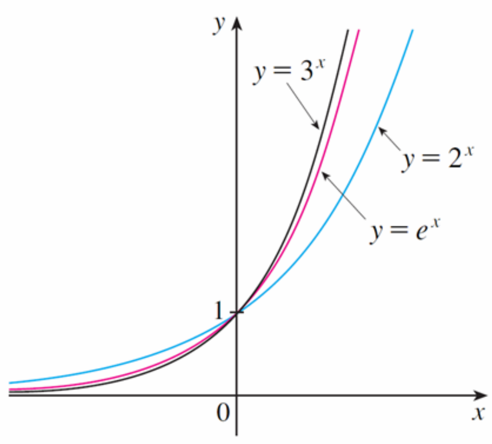


FIGURE 6

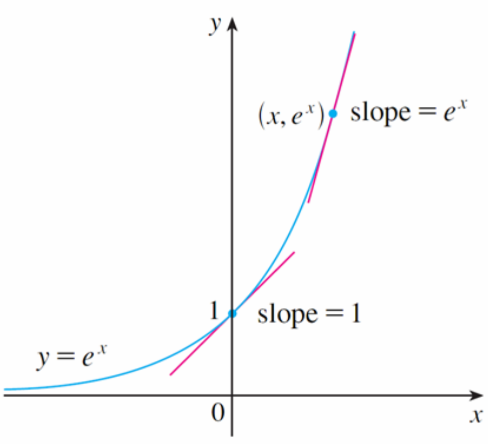


FIGURE 7

This leads to the following differentiation formula...

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

This is the ONLY function that is its own derivative

$$f(x) = e^x$$

$$\boxed{f'(x) = e^x}$$

In General...

$$\frac{d(e^u)}{dx} = e^u \bullet du$$

Differentiating Exponential Functions

$$y = e^{3x^7}$$

$u = 3x^7 \quad du = 21x^6$

$y' = e^{3x^7} \cdot 21x^6$

$y' = 21x^6 e^{3x^7}$

$$y = e^{\sin x}$$

$u = \sin x$

$du = \cos x$

$y' = e^{\sin x} \cdot \cos x$

$y' = \cos x \, e^{\sin x}$

$$y = (x^2)(e^x)$$

$y' = 2x(e^x) + x^2 e^x$

$y' = 2xe^x + x^2 e^x$

$y' = xe^x(2 + x)$

$$y = e^{\cot x^3}$$

$y' = e^{\cot x^3} \cdot -\csc^2 x^3 \cdot 3x^2$

$y' = -3x^2 \csc^2 x^3 \, e^{\cot x^3}$

Practice Exercises

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#4, 5, 6, 8, 9, 10,

Bonus:

Give that $y = \cos^{-1}(\cos^{-1} x)$, prove that

$$\frac{dy}{dx} = \frac{1}{\sin y\sqrt{1-x^2}}$$