Warm Up

Review of laws of logarithms...

Express the following as a single logarithm:

$$\frac{2}{3}\ln b^{6}-\frac{1}{2}\left[\ln b^{4}+8\ln b+6\ln\sqrt[3]{b}\right]$$

$$\frac{2}{3}\ln b^{6}-\frac{1}{2}\ln b^{4}-4\ln b-3\ln b^{1/3}$$

$$\ln b^{4}-\ln b^{2}-\ln b^{4}-\ln b$$

$$\ln\left(\frac{b^{4}}{b^{2}b^{4}b}\right)$$

$$\ln\left(\frac{1}{b^{3}}\right)$$

$$\ln b^{-3}$$

$$\boxed{-3\ln b}$$

Questions from Homework

① b) $y=\ln\left(\frac{x}{\sqrt{x^2+1}}\right)$ $u=\frac{x}{(x^2+1)^{1/2}}$ du ⇒ quotient/chain

$$y'=\frac{(x^2+1)^{1/2}}{x}\left[\frac{(x^2+1)^{1/2}(1)-x(\cancel{\tfrac{1}{2}})(x^2+1)^{-1/2}(\cancel{2}x)}{x^2+1}\right]$$

$$y'=\frac{(x^2+1)^{1/2}}{x}\left[\frac{(x^2+1)^{1/2}-x^2(x^2+1)^{-1/2}}{x^2+1}\right]$$

$$y'=\frac{\underline{(x^2+1)^{1/2}}}{x}\left[\frac{\underline{(x^2+1)^{-1/2}}\overset{1}{[\cancel{x^2}+1-\cancel{x^2}]}}{x^2+1}\right]$$

$$y'=\frac{\underline{1}}{x(x^2+1)}$$

② $y=\ln(\ln x)$ $u=\ln x$ $du=\frac{1}{x}$

$$y'=\frac{1}{\ln x}\cdot\frac{1}{x}=\frac{1}{x\ln x}$$

Questions from Homework

③ d) $g(x) = \frac{1+\log_3 x}{x}$

$$g'(x) = \frac{\cancel{x}\left(\frac{1}{\cancel{x}\ln 3}\cdot 1\right) - (1+\log_3 x)(1)}{x^2}$$

$$g'(x) = \frac{\overset{\ln 3}{\frac{1}{\ln 3}} - \overset{\ln 3}{1} - \overset{\ln 3}{\log_3 x}}{x^2 \ln 3}$$

$$g'(x) = \frac{1 - \ln 3 - \ln 3(\log_3 x)}{x^2 \ln 3}$$

Change of base formula.

$$\log_N M = \frac{\log_b M}{\log_b N} = \frac{\log M}{\log N} = \frac{\ln M}{\ln N}$$

base 10

base e

$$g'(x) = \frac{1 - \ln 3 - \cancel{\ln 3}\left(\frac{\ln x}{\cancel{\ln 3}}\right)}{x^2 \ln 3}$$

$$\boxed{g'(x) = \frac{1 - \ln 3 - \ln x}{x^2 \ln 3}}$$

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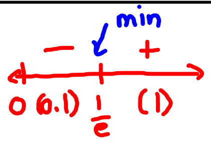
⑦ $f(x) = x\ln x$



(i) $F'(x) = \cancel{x}\left(\frac{1}{\cancel{x}}\right) + (1)\ln x$

$F'(x) = 1 + \ln x$

$1 + \ln x = 0$

$\ln x = -1$

$e^{-1} = x$

CV: $\frac{1}{e} = x$

$f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)\ln\left(\frac{1}{e}\right)$ → $\ln e^{-1}$

$= \frac{1}{e}(-1)$

$= -\frac{1}{e}$

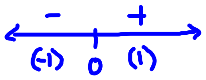
$\left(\frac{1}{e}, -\frac{1}{e}\right)$ min

(ii) $F'(x) = 1 + \ln x$

$F''(x) = 0 + \frac{1}{x}$

$F''(x) = \frac{1}{x}$

CV: $x = 0$



CU on $(0, \infty)$

CD on $(-\infty, 0)$

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⑧ $f(x) = x(\ln x)^2$

$F'(x) = \cancel{x}(2)(\ln x)\left(\frac{1}{\cancel{x}}\right) + (1)(\ln x)^2$

$F'(x) = 2\ln x + (\ln x)^2$

$F'(x) = \ln x[2 + \ln x]$

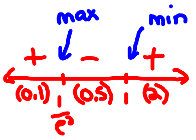
$0 = \ln x[2 + \ln x]$

CV: $\ln x = 0$ | $2 + \ln x = 0$

$e^0 = x$ | $\ln x = -2$

$1 = x$ | $e^{-2} = x$

$\frac{1}{e^2} = x$



$f\left(\frac{1}{e^2}\right) = \frac{4}{e^2}$ max

$f(1) = 0$ min

Questions from Homework

⑥ $\ln(x+y) = y - 1$

$$\frac{1}{(x+y)} \cdot \left(1 + \frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\cancel{(x+y)}\,\frac{1 + \frac{dy}{dx}}{\cancel{x+y}} = \frac{dy}{dx}(x+y)$$

$$1 + \frac{dy}{dx} = x\frac{dy}{dx} + y\frac{dy}{dx}$$

$$1 = x\frac{dy}{dx} + y\frac{dy}{dx} - \frac{dy}{dx}$$

$$\frac{1}{x+y-1} = \frac{\frac{dy}{dx}\cancel{(x+y-1)}}{\cancel{x+y-1}}$$

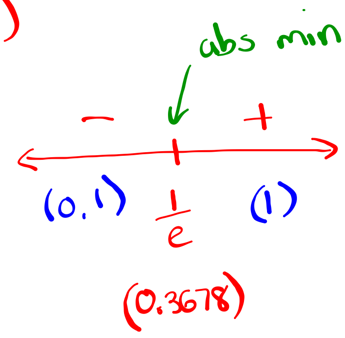
$$\boxed{\frac{1}{x+y-1} = \frac{dy}{dx}}$$

Questions from Homework

⑦ a) $y = x\ln x$

$\frac{dy}{dx} = \ln x + x\left(\frac{1}{x}\cdot 1\right)$

$\frac{dy}{dx} = \ln x + 1$



$0 = \ln x + 1$

$-1 = \ln x$ (log form)

abs min @ $x = \frac{1}{e}$

$e^{-1} = x$ (exp form)

CV: → $\frac{1}{e} = x$

$y = x\underbrace{\ln x}$

$y = \left(\frac{1}{e}\right)\underbrace{\ln\left(\frac{1}{e}\right)}$

$\ln\left(\frac{1}{e}\right) = ?$

$\ln e^{-1} = ?$

$\log_e e^{-1} = ?$

$\log_e e^{-1} = \underline{\underline{-1}}$

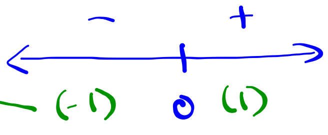
$y = \left(\frac{1}{e}\right)\underline{(-1)}$

$y = -\frac{1}{e}$

$\boxed{\left(\frac{1}{e}, -\frac{1}{e}\right)}$

b) $\frac{dy}{dx} = \ln x + 1$

$\frac{d^2y}{dx^2} = \frac{1}{x}$ ←



CU on $(0,\infty)$

CV: $x = 0$

Logarithmic Differentiation

A differentiation process that requires taking the logarithm of both sides before differentiating. (ln)

$$\boxed{\frac{d}{dx}\ln u = \frac{du}{u}}$$

This process will be used in TWO circumstances:

I. Simplifying messy products and quotients

What would it involve to differentiate the following?

$$y=\frac{\left(x^2-1\right)^5\sqrt{2x+9}\left(5x^3+2\right)^8}{\left(10x-1\right)\sqrt{5-x^7}}$$

- Quotient rule, multiple product rules and chain rules...

This would be possible but it would be easier to differentiate a group of terms added and subtracted rather than multiplied and divided

Laws of logarithms will do exactly that...turn this mess into a addition and subtraction of terms.

$$y=\frac{\left(x^2-1\right)^5\sqrt{2x+9}\left(5x^3+2\right)^8}{\left(10x-1\right)\sqrt{5-x^7}}$$

$$\ln y = \ln\left[\frac{(x^2-1)^5(2x+9)^{1/2}(5x^3+2)^8}{(10x-1)(5-x^7)^{1/2}}\right]$$

$$\ln y = \ln(x^2-1)^{5} + \ln(2x+9)^{1/2} + \ln(5x^3+2)^{8} - \ln(10x-1) - \ln(5-x^7)^{1/2}$$

$$\ln y = 5\ln(x^2-1) + \frac{1}{2}\ln(2x+9) + 8\ln(5x^3+2) - \ln(10x-1) - \frac{1}{2}\ln(5-x^7)$$

$$\frac{y'}{y} = 5\left(\frac{2x}{x^2-1}\right) + \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{2x+9}\right) + 8\left(\frac{15x^2}{5x^3+2}\right) - \left(\frac{10}{10x-1}\right) - \frac{1}{2}\left(\frac{-7x^6}{5-x^7}\right)$$

$$\cancel{y}\cdot\frac{y'}{\cancel{y}} = \left(\frac{10x}{x^2-1} + \frac{1}{2x+9} + \frac{120x^2}{5x^3+2} - \frac{10}{10x-1} + \frac{7x^6}{10-2x^7}\right)\cdot y$$

$$y' = \left[\frac{10x}{x^2-1} + \frac{1}{2x+9} + \frac{120x^2}{5x^3+2} - \frac{10}{10x-1} + \frac{7x^6}{10-2x^7}\right]\left[\frac{(x^2-1)^5(\sqrt{2x+9})(5x^3+2)^8}{(10x-1)(\sqrt{5-x^7})}\right]$$

Steps in Logarithmic Differentiation

1. Take logarithms of both sides of an equation. (ln)
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'

Use Logarithmic Differentiation to Differentiate the following:

$$y = \frac{e^x\sqrt{x^2+1}}{\left(x^2+2\right)^3}$$

$$\ln y = \ln\left[\frac{e^x(x^2+1)^{1/2}}{(x^2+2)^3}\right]$$

$$\ln y = \ln e^x + \ln(x^2+1)^{1/2} - \ln(x^2+2)^3$$

$$\ln e = \underline{\underline{1}}$$

$$\ln y = x\underline{\underline{\ln e}} + \frac{1}{2}\ln(x^2+1) - 3\ln(x^2+2)$$

$$\ln y = x + \frac{1}{2}\ln(x^2+1) - 3\ln(x^2+2)$$

$$\frac{y'}{y} = 1 + \frac{1}{2}\left(\frac{2x}{x^2+1}\right) - 3\left(\frac{2x}{x^2+2}\right)$$

$$\cancel{y}\cdot \quad \frac{y'}{\cancel{y}} = \left(1 + \frac{x}{x^2+1} - \frac{6x}{x^2+2}\right)\cdot y$$

$$y' = \left[1 + \frac{x}{x^2+1} - \frac{6x}{x^2+2}\right]\left[\frac{e^x\sqrt{x^2+1}}{(x^2+2)^3}\right]$$

II. Base and exponent both variables

Have a look at this example:

$$y = x^{x^5}$$

- Does not fit either the power rule or the rules for an exponential function

...What can be done to help this crazy situation??

Of Course...take the logarithm of both sides!!

$$y = x^{x^5}$$

$$\ln y = \ln x^{x^5}$$

$$\ln y = x^5 \ln x$$ ← product rule

$$\frac{y'}{y} = 5x^4 \ln x + x^5\left(\frac{1}{x}\right)$$

$$\cancel{y}\cdot \quad \frac{y'}{\cancel{y}} = 5x^4 \ln x + x^4 \cdot y$$

$$y' = \left(5x^4 \ln x + x^4\right)\left(x^{x^5}\right)$$

Example:

Differentiate: $y = \left(\ln x^5\right)^{\cos x}$

Practice Questions...

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#1 b, d, e

#2 b, c, e

#3