Exercise 7.6

Differentiating Inverse Trig Functions:

$$u = \frac{\cos x}{1+\sin x} \qquad \frac{du}{dx} = \frac{-\sin x(1+\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$$

$$\frac{du}{dx} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

① i) $y = \sin^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

$$y' = \frac{1}{\sqrt{1-\left(\frac{\cos x}{1+\sin x}\right)^2}} \cdot \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$y' = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2\sqrt{1 - \frac{(\cos x)^2}{(1+\sin x)^2}}}$$

$$y' = \frac{-(\sin x + \sin^2 x + \cos^2 x)}{(1+\sin x)^2\sqrt{\left(1 - \frac{\cos^2 x}{(1+\sin x)^2}\right)}}$$

pythagorean identity $\sin^2 x + \cos^2 x = 1$

$$y' = \frac{-(\cancel{\sin x + 1})}{(1+\sin x)^2\sqrt{1 - \frac{\cos^2 x}{(1+\sin x)^2}}}$$

$$y' = \frac{-1}{\sqrt{(1+\sin x)^2\left(1 - \frac{\cos^2 x}{(1+\sin x)^2}\right)}}$$

$$y' = \frac{-1}{\sqrt{(1+\sin x)^2 - \cos^2 x}}$$

$(1+\sin x)(1+\sin x)$

$1 + \sin x + \sin x + \sin^2 x$

$$y' = \frac{-1}{\sqrt{1 + 2\sin x + \sin^2 x - \cos^2 x}}$$

$$y' = \frac{-1}{\sqrt{\sin^2 x + \cancel{\cos^2 x} + 2\sin x + \sin^2 x - \cancel{\cos^2 x}}}$$

$$y' = \frac{-1}{\sqrt{2\sin^2 x + 2\sin x}}$$

$$y' = \frac{-1}{\sqrt{2\sin x(\sin x + 1)}}$$

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① n)
$$y = \frac{\sqrt{1-x^2}}{x} + \sin^{-1}\sqrt{x}$$

$$y = \frac{(1-x^2)^{1/2}}{x} + \sin^{-1}\sqrt{x}$$

$$\frac{dy}{dx} = \frac{\frac{1}{\cancel{2}}(1-x^2)^{-1/2}\overset{(-x)}{\cancel{(-2x)}}(x) - 1(1-x^2)^{1/2}}{x^2} + \frac{1}{\sqrt{1-x^2}} \cdot 1$$

$$\frac{dy}{dx} = \frac{-x^2(1-x^2)^{-1/2} - 1(1-x^2)^{1/2}}{x^2} + \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{(1-x^2)^{-1/2}\left[\overset{-x^2-1+x^2}{-x^2 - (1-x^2)}\right]}{x^2} + \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-1}{x^2\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-1}{x^2\sqrt{1-x^2}} + \frac{x^2}{x^2\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{x^2-1}{x^2\sqrt{1-x^2}}$$

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① p) $y = \sin^{-1}x + \cos^{-1}\sqrt{1-x^2}$

$u = (1-x^2)^{1/2}$

$\frac{du}{dx} = \frac{1}{\cancel{2}}(1-x^2)^{-1/2}(-\cancel{2}x)$

$\frac{du}{dx} = \frac{-x}{\sqrt{1-x^2}}$

$$y' = \frac{1}{\sqrt{1-x^2}} \cdot 1 + \frac{-1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{-x}{\sqrt{1-x^2}}$$

$$y' = \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}\sqrt{1-(1-x^2)}}$$

$$y' = \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}\cdot\sqrt{x^2}}$$

$$y' = \frac{1}{\sqrt{1-x^2}} + \frac{\cancel{x}}{\sqrt{1-x^2}\cdot\cancel{x}}$$

$$y' = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$

$$\boxed{y' = \frac{2}{\sqrt{1-x^2}}}$$

Series + Sequence

① $a = 90000$

$t_6 = 158\,610.75$

$n = 6$

$r = ?$

$t_n = ar^{n-1}$

$$\frac{158\,610.75}{90000} = \frac{\cancel{90000}\, r^{6-1}}{\cancel{90000}}$$

$1.7623\ldots = r^5$

$1.12 = r$

$AROI = 100(1.12 - 1)$

$= 12\%$

Series + Sequence

② $a = 105$ $\quad t_n = a + (n-1)d$

$d = 90$ $\quad t_6 = 105 + (6-1)(90)$

$t_6 = ?$ $\quad t_6 = 105 + 5(90)$

$t_6 = 105 + 450$

$t_6 = \$555$

Series + Sequence

3 $a = 5 \times 3$

$a = 15$

$r = 3$

a) Cycle 1 = a = 15

b) $t_n = ar^{n-1}$

$t_n = 15(3)^{n-1}$

c) $\frac{32805}{15} = \frac{\cancel{15}(3)^{n-1}}{\cancel{15}}$

$2187 = 3^{n-1}$

$\cancel{3}^7 = \cancel{3}^{n-1}$

$7 = n - 1$

$\boxed{8 = n}$

Cycle 8

d) $S_n = \frac{a(r^n - 1)}{r-1}$

$S_4 = \frac{15(3^4 - 1)}{3-1}$

$S_4 = \frac{15(\cancel{80})}{\cancel{2}}$ 40

$\boxed{S_4 = 600}$

Series + Sequence:

4 a) $\lim_{n\to\infty} \frac{1-2n^4}{3n^4+5} = \frac{-2}{3}$

b) $\lim_{n\to\infty} \frac{n^4-1}{n^6+7} = 0$

c) $\lim_{n\to\infty} \frac{n^2-7n+12}{n+2} = \text{DNE}$

Series + Sequence:

5 a) $\sum_{n=1}^{6} n^2 - 1 = 0 + 3 + 8 + 15 + 24 + 35$

$= 85$

b) $\sum_{n=1}^{\infty} (5)\left(\frac{1}{4}\right)^{n-1}$

infinite geometric

$a = 5$
$r = \frac{1}{4}$

$S_n = \frac{a}{1-r}$

$S_n = \frac{5}{1-\frac{1}{4}}$

$S_n = \frac{5}{\frac{3}{4}}$

$S_n = 5 \cdot \frac{4}{3}$

$S_n = \frac{20}{3}$

c) $1 + 4 + 7 + \ldots + 55$

$a = 1$
$d = 3$
$t_n = 55$

(i) Find n:

$t_n = a + (n-1)d$

$55 = 1 + (n-1)3$

$\frac{54}{3} = \frac{3(n-1)}{3}$

$18 = n - 1$

$19 = n$

(ii) Find S_{19}

$S_n = \frac{n}{2}(a + t_n)$

$S_{19} = \frac{19}{2}(1 + 55)$

$S_{19} = \frac{19}{\cancel{2}}(\cancel{56})$ 28

$\boxed{S_{19} = 532}$

d) $S_8 = 2 - 6 + 18 \ldots$

$a = 2$
$r = -3$
$n = 8$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$S_8 = \frac{2((-3)^8 - 1)}{-3-1}$

$S_8 = \frac{2(6560)}{-4}$

$\boxed{S_8 = -3280}$

Series + Sequence:

Eliminate "a" by dividing

6 $t_3 = 5$ | $t_7 = 405$

$t_3 = ar^{3-1}$ | $t_7 = ar^{7-1}$

$t_3 = ar^2$ | $t_7 = ar^6$

$5 = ar^2$ | $405 = ar^6$

$$\frac{405 = ar^6}{5 = ar^2}$$

$81 = r^4$

$\underline{\underline{\pm 3 = r}}$

$ar^2 = 5$

$a(3)^2 = 5$

$9a = 5$

$\underline{\underline{a = \frac{5}{9}}}$

$$\boxed{t_n = \frac{5}{9}(-3)^{n-1} \text{ or } t_n = \frac{5}{9}(3)^{n-1}}$$

Series + Sequence:

7) $t_7 = 19$

$t_n = a + (n-1)d$

$t_7 = a + (7-1)d$

$t_7 = a + 6d$

$19 = a + 6d$

$S_{10} = 130$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{10} = \frac{10}{2}[2a + (10-1)d]$

$S_{10} = 5(2a + 9d)$

$S_{10} = 10a + 45d$

$130 = 10a + 45d$

Eliminate "a" by subtracting

$190 = 10a + 60d$

$(-)\ 130 = 10a + 45d$

$\frac{60}{15} = \frac{15d}{15}$

$\boxed{4 = d}$

Find a:

$19 = a + 6d$

$19 = a + 6(4)$

$19 = a + 24$

$\boxed{-5 = a}$

$\boxed{-5, -1, 3, \ldots}$

Permutations + Combinations

binomial theore

- used to expand $(x + y)^n$, $n \in N$
- each term has the f $_nC_k(x)^{n-k}(y)^k$, where $k + 1$ is the term number

You can use the **binomial theorem** to expand any power of a binomial expression.

$$(x + y)^n = {}_nC_0(x)^n(y)^0 + {}_nC_1(x)^{n-1}(y)^1 + {}_nC_2(x)^{n-2}(y)^2 + \cdots + {}_nC_{n-1}(x)^1(y)^{n-1} + {}_nC_n(x)^0(y)^n$$

In this chapter, all binomial expansions will be written in descending order of the exponent of the first term in the binomial.

The following are some important observations about the expansion of $(x + y)^n$, where x and y represent the terms of the binomial and $n \in N$:

- the expansion contains $n + 1$ terms
- the number of objects, k, selected in the combination $_nC_k$ can be taken to match the number of factors of the second variable selected; that is, it is the same as the exponent on the second variable
- the general term, t_{k+1}, has the form

$${}_nC_k(x)^{n-k}(y)^k$$

the same

- the sum of the exponents in any term of the expansion is n

15 $\left(y - \frac{2}{y^2}\right)^5$

$${}_5C_0(y)^5\left(\frac{-2}{y^2}\right)^0 + {}_5C_1(y)^4\left(\frac{-2}{y^2}\right)^1 + {}_5C_2(y)^3\left(\frac{-2}{y^2}\right)^2 + {}_5C_3(y)^2\left(\frac{-2}{y^2}\right)^3 + {}_5C_4(y)^1\left(\frac{-2}{y^2}\right)^4 + {}_5C_5(y)^0\left(\frac{-2}{y^2}\right)^5$$

$$1(y^5)(1) + 5(y^4)\left(\frac{-2}{y^2}\right) + 10(y^3)\left(\frac{4}{y^4}\right) + 10(y^2)\left(\frac{-8}{y^6}\right) + 5(y)\left(\frac{16}{y^8}\right) + 1(1)\left(\frac{-32}{y^{10}}\right)$$

$$y^5 - \frac{10y^4}{y^2} + \frac{40y^3}{y^4} - \frac{80y^2}{y^6} + \frac{80y}{y^8} - \frac{32}{y^{10}}$$

$$y^5 - 10y^2 + \frac{40}{y} - \frac{80}{y^4} + \frac{80}{y^7} - \frac{32}{y^{10}}$$

Find the term in the expansion of

$(2x+y)^8$ that contains x^5y^3

${}_8C_3\,(2x)^5(y)^3$

$(56)(32x^5)(y^3)$

$1792\,x^5y^3$

Permutations + Combinations

5 $(2x^2+3y)^7 \rightarrow 3^{rd}$ term

$_7C_2(2x^2)^5(3y)^2$

$21(32x^{10})(9y^2)$

$\boxed{6048x^{10}y^2}$

Permutations + Combinations

② $\left(x^2-\frac{x}{2}\right)^5$

$${}_5C_0(x^2)^5\left(-\frac{x}{2}\right)^0+{}_5C_1(x^2)^4\left(-\frac{x}{2}\right)^1+{}_5C_2(x^2)^3\left(-\frac{x}{2}\right)^2+{}_5C_3(x^2)^2\left(-\frac{x}{2}\right)^3+{}_5C_4(x^2)^1\left(-\frac{x}{2}\right)^4+{}_5C_5(x^2)^0\left(-\frac{x}{2}\right)^5$$

$$1(x^{10})(1)+5(x^8)\left(-\frac{x}{2}\right)+10(x^6)\left(\frac{x^2}{4}\right)+10(x^4)\left(-\frac{x^3}{8}\right)+5(x^2)\left(\frac{x^4}{16}\right)+1(1)\left(-\frac{x^5}{32}\right)$$

$$x^{10}-\frac{5}{2}x^9+\frac{5}{2}x^8-\frac{5}{4}x^7+\frac{5}{16}x^6-\frac{x^5}{32}$$

Perm/Comb

② $\left(x^2 - \frac{x}{3}\right)^3$ $a = x^2$ $b = -\frac{x}{3}$ $n = 3$

$${}_3C_0(x^2)^3\left(-\frac{x}{3}\right)^0 + {}_3C_1(x^2)^2\left(-\frac{x}{3}\right)^1 + {}_3C_2(x^2)^1\left(-\frac{x}{3}\right)^2 + {}_3C_3(x^2)^0\left(-\frac{x}{3}\right)^3$$

$$1(x^6)(1) + 3(x^4)\left(-\frac{x}{3}\right) + 3(x^2)\left(\frac{x^2}{9}\right) + 1(1)\left(\frac{-x^3}{27}\right)$$

$$x^6 - \frac{3x^5}{3} + \frac{3x^4}{9} - \frac{1x^3}{27}$$

$$\boxed{x^6 - x^5 + \frac{1}{3}x^4 - \frac{1}{27}x^3}$$

Permutations + Combinations

Expand: $\left(x-\frac{1}{2}\right)^{4}$

$${}_4C_0(x)^4\left(-\frac{1}{2}\right)^0+{}_4C_1(x)^3\left(-\frac{1}{2}\right)^1+{}_4C_2(x)^2\left(-\frac{1}{2}\right)^2+{}_4C_3(x)^1\left(-\frac{1}{2}\right)^3+{}_4C_4(x)^0\left(-\frac{1}{2}\right)^4$$

$$(1)(x^4)(1)+(4)(x^3)\left(-\frac{1}{2}\right)+(6)(x^2)\left(\frac{1}{4}\right)+(4)(x)\left(-\frac{1}{8}\right)+(1)(1)\left(\frac{1}{16}\right)$$

$$x^4-\frac{4x^3}{2}+\frac{6x^2}{4}-\frac{4x}{8}+\frac{1}{16}$$

$$\boxed{x^4-2x^3+\frac{3}{2}x^2-\frac{1}{2}x+\frac{1}{16}}$$

Limits

$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$

$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$

Limits

① a) $\lim_{x \to 7} \frac{\sqrt{x+9}-4}{x-7} \cdot \frac{\sqrt{x+9}+4}{\sqrt{x+9}+4}$

$\lim_{x \to 7} \frac{x+9-16}{(x-7)(\sqrt{x+9}+4)}$

$\lim_{x \to 7} \frac{\cancel{(x-7)}}{\cancel{(x-7)}(\sqrt{x+9}+4)} = \frac{1}{8}$

b) $\lim_{x \to 4} \frac{16-x^2}{2x^2-11x+12}$ ← Diff of squares ← Hard trinomial

$-8 + -3 = -11$

$-8 \times -3 = 24$

$= \lim_{x \to 4} \frac{(4-x)(4+x)}{(x-\frac{8}{2})(x-\frac{3}{2})}$

$= \lim_{x \to 4} \frac{\overset{-1}{\cancel{(4-x)}}(4+x)}{\overset{1}{\cancel{(x-4)}}(2x-3)} = \frac{-8}{5}$

Limits

① c) $$\lim_{a\to b} \frac{(a+2b)^2-9b^2}{a-b}$$

$$\lim_{a\to b} \frac{(a+2b+3b)(a+2b-3b)}{(a-b)}$$

$$\lim_{a\to b} \frac{(a+5b)\cancel{(a-b)}}{\cancel{(a-b)}} = 6b$$

d) $$\lim_{x\to 0} \frac{x^3+1}{x^2+3x+2} = \frac{1}{2}$$

e) $$\lim_{x\to\infty} \frac{9-x^4}{(3x^2-2)^2}$$

$$= \lim_{x\to\infty} \frac{9-x^4}{9x^4-12x^2+4}$$

$$= \lim_{x\to\infty} \frac{\frac{9}{x^4}-\frac{x^4}{x^4}}{\frac{9x^4}{x^4}-\frac{12x^2}{x^4}+\frac{4}{x^4}}$$

$$= \lim_{x\to\infty} \frac{9/x^4 - 1}{9 - 12/x^2 + 4/x^4} = -\frac{1}{9}$$

approaching 0.

① f) $\lim_{x\to 0} \dfrac{\cancel{5(x+5)}\frac{1}{\cancel{x+5}} - \frac{1}{\cancel{5}}\cancel{5}(x+5)}{(x^2+5x)(5(x+5))}$ CD: $5(x+5)$

$\lim_{x\to 0} \dfrac{5-1(x+5)}{x(x+5)(5(x+5))}$

$\lim_{x\to 0} \dfrac{5-x-5}{5x(x+5)^2}$

$\lim_{x\to 0} \dfrac{-\cancel{x}}{5\cancel{x}(x+5)^2} = \dfrac{-1}{5(25)} = \dfrac{-1}{125}$

g) $\lim_{x\to b} \dfrac{x^2-b^2}{x^8-b^8}$

$\lim_{x\to b} \dfrac{(x^2-b^2)}{(x^4+b^4)(x^4-b^4)}$

$\lim_{x\to b} \dfrac{\cancel{(x^2-b^2)}}{(x^4+b^4)(x^2+b^2)\cancel{(x^2-b^2)}} = \dfrac{1}{(2b^4)(2b^2)} = \dfrac{1}{4b^6}$

h) $\lim_{h\to -4} \dfrac{8+(2+h)^3}{h+4}$ ← sum of cubes

$= \lim_{h\to -4} \dfrac{[2+(2+h)][4-2(2+h)+(2+h)^2]}{h+4}$

$= \lim_{h\to -4} \dfrac{\cancel{(h+4)}[4-2(2+h)+(2+h)^2]}{\cancel{(h+4)}}$

$= 4+4+4$

$= 12$

Limits

(2) Let $f(x) = \begin{cases} -x^2+2, & \text{if } x<1 \\ 1, & \text{if } x=1 \\ -2x+3, & \text{if } 1<x\leq 3 \\ -3, & \text{if } x>3 \end{cases}$

$-x^2+2$

x	y
∘ 1	1
0	2
-1	1
-2	-2

1

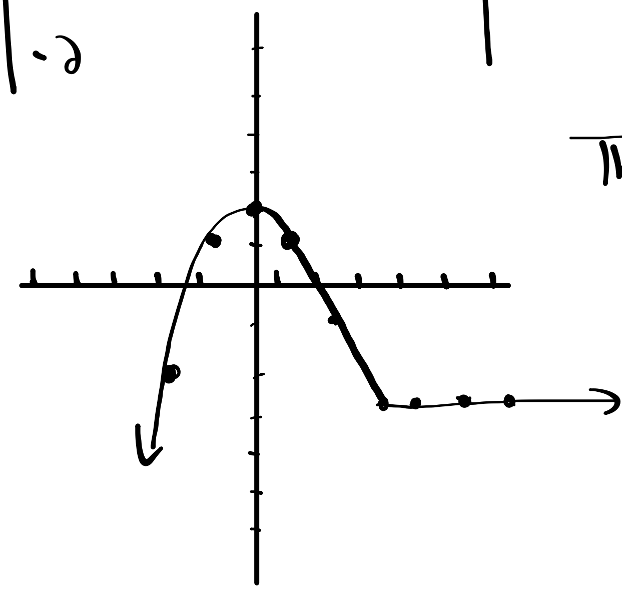
x	y
• 1	1

$-2x+3$

x	y
∘ 1	1
2	-1
• 3	-3

-3

x	y
∘ 3	-3
4	-3
5	-3
6	-3



The function is continuous

Limits

$$\lim_{x\to 0} \frac{\cancel{3(x+3)}\,\frac{1}{\cancel{x+3}} - \frac{1}{\cancel{3}}\,\cancel{3}(x+3)}{x \cdot 3(x+3)}$$

CD: $3(x+3)$

$$\lim_{x\to 0} \frac{3-(x+3)}{3x(x+3)}$$

$$\lim_{x\to 0} \frac{3-x-3}{3x(x+3)}$$

$$\lim_{x\to 0} \frac{-\cancel{x}}{3\cancel{x}(x+3)} = \frac{-1}{3(3)} = -\frac{1}{9}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

① a) $f(x) = \sqrt{x-5}$

ⓘ $f(x+h) = \sqrt{x+h-5}$

ⓘⓘ $$f'(x) = \lim_{h\to 0} \frac{\left(\sqrt{x+h-5} - \sqrt{x-5}\right)\left(\sqrt{x+h-5} + \sqrt{x-5}\right)}{h\left(\sqrt{x+h-5} + \sqrt{x-5}\right)}$$

$$= \lim_{h\to 0} \frac{x+h-5-(x-5)}{h\left(\sqrt{x+h-5} + \sqrt{x-5}\right)}$$

$$= \lim_{h\to 0} \frac{x+h-5-x+5}{h\left(\sqrt{x+h-5} + \sqrt{x-5}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+\underline{h}-5} + \sqrt{x-5}\right)}$$

$$= \frac{1}{2\sqrt{x-5}}$$

Differentiation

⑤ $y=(x^2-3)^8$ @ $x=2$

(i) Find y:

$y=(2^2-3)^8$

$y=(4-3)^8$

$y=\underline{1}$

(ii) Find y':

$y=(x^2-3)^8$

$y'=8(x^2-3)^7(2x)$

$y'=16x(x^2-3)^7$

(iii) Find slope (m):

$y'=16(2)[(2)^2-3]^7$

$y'=32[4-3]^7$

$y'=32$ ← $m=32$

(iv): $y-y_1=m(x-x_1)$

$y-1=32(x-2)$

$y-1=32x-64$

$\boxed{y=32x-63}$

or $\boxed{32x-y-63=0}$

⑦ a) $f(x)=\left(\dfrac{2x+1}{x-1}\right)^5$

$f'(x)=5\left(\dfrac{2x+1}{x-1}\right)^4\left[\dfrac{2(x-1)-1(2x+1)}{(x-1)^2}\right]$

$2x-2 \quad -2x-1$

$f'(x)=5\dfrac{(2x+1)^4}{(x-1)^4}\cdot\dfrac{-3}{(x-1)^2}$

$f'(x)=\dfrac{-15(2x+1)^4}{(x-1)^6}$

Differentiation

Product: $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$

$$y = \underbrace{(3x^2-5)}_{f(x)}\underbrace{(4x^3+3x)}_{g(x)}$$

$$y' = 6x(4x^3+3x) + (3x^2-5)(12x^2+3)$$

$$y' = 24x^4 + 18x^2 + 36x^4 + 9x^2 - 60x^2 - 15$$

$$\boxed{y' = 60x^4 - 33x^2 - 15}$$

Quotient: $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

$$y = \frac{x^2+2 \leftarrow f(x)}{3x+5 \leftarrow g(x)}$$

$$y' = \frac{2x(3x+5) - 3(x^2+2)}{(3x+5)^2}$$

$$y' = \frac{6x^2 + 10x - 3x^2 - 6}{(3x+5)^2} = \boxed{\frac{3x^2+10x-6}{(3x+5)^2}}$$

Chain: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

$$y = \sqrt{4x^3-6x} = (4x^3-6x)^{1/2}$$

$$y' = \frac{1}{2}(4x^3-6x)^{-1/2}(12x^2-6)$$

$$y' = \frac{12x^2-6}{2(4x^3-6x)^{1/2}} = \frac{\overset{3}{\cancel{6}}(2x^2-1)}{\cancel{2}\sqrt{4x^3-6x}} = \boxed{\frac{3(2x^2-1)}{\sqrt{4x^3-6x}}}$$

$$\frac{6x^2-3}{\sqrt{4x^3-6x}}$$

Combo:

$$y = (3x^2+5)^3\sqrt{4x-2} = \underbrace{(3x^2+5)^3}_{f(x)}\underbrace{(4x-2)^{1/2}}_{g(x)}$$

$$y' = \underbrace{3(3x^2+5)^2(6x)}_{f'(x)}\underbrace{(4x-2)^{1/2}}_{g(x)} + \underbrace{(3x^2+5)^3}_{f(x)}\underbrace{\left(\frac{1}{2}\right)(4x-2)^{-1/2}(4)}_{g'(x)}$$

$$y' = 18x(3x^2+5)^2(4x-2)^{1/2} + 2(3x^2+5)^3(4x-2)^{-1/2}$$

$$\cdot\ 36x^2 - 18x + 3x^2 + 5$$

$$y' = 2(3x^2+5)^2(4x-2)^{-1/2}\left[9x(4x-2) + (3x^2+5)\right]$$

$$y' = 2(3x^2+5)^2(4x-2)^{-1/2}(39x^2-18x+5)$$

$$y' = \frac{2(3x^2+5)^2(39x^2-18x+5)}{\sqrt{4x-2}}$$

$$\frac{(3x^2+5)^2}{(3x^2+5)^2} = (3x^2+5)^0 = 1 \quad \Bigg| \quad \frac{(4x-2)^{1/2}}{(4x-2)^{-1/2}} = (4x-2)^{1/2-(-1/2)} = (4x-2)^1$$

Differentiation

$-\frac{1}{2}\times 3 = \frac{-3}{2}$

$-\frac{1}{2}-1$

$-\frac{1}{2}-\frac{2}{2} = \frac{-3}{2}$

② b) $f(x) = \frac{3}{\sqrt{x}} = \frac{3}{x^{1/2}} = 3x^{-1/2}$

$f'(x) = -\frac{3}{2}x^{-3/2} = \frac{-3}{2x^{3/2}}$

④ b) $y = \frac{\sqrt{x}}{3+x^2} = \frac{x^{1/2}}{3+x^2}$ $\begin{matrix} f \\ g \end{matrix}$

$\frac{f'g - fg'}{(g)^2}$

$y' = \frac{\frac{1}{2}x^{-1/2}(3+x^2) - x^{1/2}(2x)}{(3+x^2)^2}$

$x^{1/2} \cdot x^1$

$x^{1/2+1}$

$x^{1/2+2/2}$

$x^{3/2}$

$y' = \frac{\frac{1}{2x^{1/2}}(3+x^2) - 2x^{3/2}}{(3+x^2)^2}$

$y' = \frac{\cancel{2x^{1/2}}\frac{(3+x^2)}{\cancel{2x^{1/2}}} - 2x^{3/2} \cdot 2x^{1/2}}{2x^{1/2}(3+x^2)^2}$

Complex Fraction:

CD: $2x^{1/2}$

$2x^{3/2} \cdot 2x^{1/2}$

$4x^{3/2+1/2} = 4x^{4/2}$

$y' = \frac{3+x^2-4x^2}{2\sqrt{x}(3+x^2)^2}$

$y' = \frac{3-3x^2}{2\sqrt{x}(3+x^2)^2}$

⑥ b) $y = \frac{16}{\sqrt{x-1}} = \frac{16}{(x-1)^{1/2}} = 16(x-1)^{-1/2}$

$y' = -8(x-1)^{-3/2}(1) = -8(x-1)^{-3/2} = \frac{-8}{(x-1)^{3/2}}$

$= \frac{-8}{\sqrt{(x-1)^3}}$

Differentiation

$f(x) = \boxed{3x^2 + 2x - 7}$

$f(x+h) = 3(x+h)^2 + 2(x+h) - 7$

$= 3(x^2 + 2xh + h^2) + 2x + 2h - 7$

$= \boxed{3x^2 + 6xh + 3h^2 + 2x + 2h - 7}$

$$f'(x) = \lim_{h \to 0} \frac{\boxed{f(x+h)} - \boxed{f(x)}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 \cancel{+ 2x} + 2h \cancel{- 7} - (\cancel{3x^2} \cancel{+ 2x} \cancel{- 7})}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{6xh + 3h^2 + 2h}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{h}(6x + 3h + 2)}{\cancel{h}} = \boxed{6x + 2}$$

Differentiation

$$f(x) = \frac{(3x^2+5)^3}{\sqrt{2x-7}} \qquad \frac{f'g - fg'}{g^2}$$

$$f'(x) = \frac{\overbrace{3(3x^2+5)^2(6x)}^{f'}\overbrace{(2x-7)^{1/2}}^{g} - \overbrace{(3x^2+5)^3}^{f}\overbrace{\left(\tfrac{1}{2}\right)(2x-7)^{-1/2}(2)}^{g'}}{\left[\sqrt{2x-7}\right]^2}$$

$$f'(x) = \frac{18x(3x^2+5)^2(2x-7)^{1/2} - (3x^2+5)^3(2x-7)^{-1/2}}{2x-7}$$ ← Factor

$$36x^2 - 126x - 3x^2 - 5$$

$$f'(x) = \frac{(3x^2+5)^2(2x-7)^{-1/2}\left[18x(2x-7) - (3x^2+5)\right]}{(2x-7)}$$

$$f'(x) = \frac{(3x^2+5)^2(33x^2-126x-5)}{(2x-7)^{3/2}}$$

Differentiation

3 $y = \sqrt[7]{2x^2 + \sqrt{x^7 - 8x\sqrt{3-x}}} = \left[2x^2 + \left(x^7 - 8x(3-x)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]^{\frac{1}{7}}$

$y' = \frac{1}{7}\left[2x^2 + \left(x^7 - 8x(3-x)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]^{-\frac{6}{7}}\left[4x + \frac{1}{2}\left(x^7 - 8x(3-x)^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(7x^6 - \left(8(3-x)^{\frac{1}{2}} + 8x\left(\frac{1}{2}\right)(3-x)^{-\frac{1}{2}}(-1)\right)\right)\right]$

Find the point (x,y) on the curve $y = x^2+5x$ where the slope of the tangent line equals 9

(i) Find derivative

$y = x^2+5x$

$y' = 2x+5$

(ii) Solve for x:

$y' = 2x+5$

$9 = 2x+5$

$4 = 2x$

$2 = x$

(iii) Solve for y:

$y = x^2+5x$

$y = (2)^2+5(2)$

$y = 4+10$

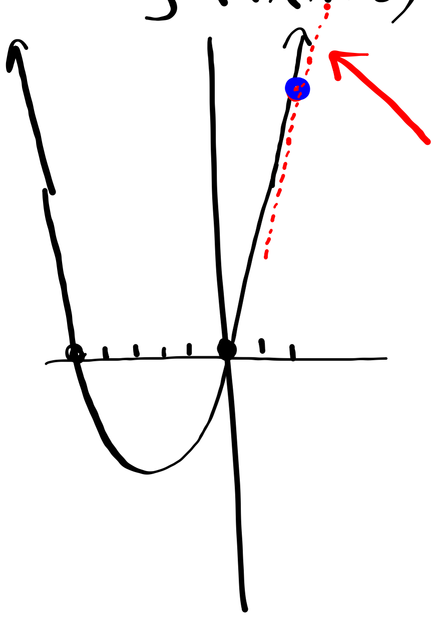
$y = 14$

(iv) Point:

$(2,14)$

$y = x^2+5x$

$y = (x)(x+5)$



The slope of that tangent is 9

Derivatives Exam Review:

④ b) $y = \frac{\sqrt{x}}{3+x^2} = \frac{x^{1/2}}{(3+x^2)}$

$$y' = \frac{(3+x^2)\left(\frac{1}{2}x^{-1/2}\right) - x^{1/2}(2x)}{(3+x^2)^2}$$

$$y' = \frac{2 \cdot \frac{3}{2}x^{-1/2} + 2 \cdot \frac{1}{2}x^{3/2} - 2 \cdot 2x^{3/2}}{2(3+x^2)^2}$$

$$y' = \frac{3x^{-1/2} + x^{3/2} - 4x^{3/2}}{2(3+x^2)^2}$$

$$y' = \frac{x^{-1/2}(3 + x^2 - 4x^2)}{2(3+x^2)^2}$$

$$y' = \frac{3-3x^2}{2x^{1/2}(3+x^2)^2} = \frac{3(1-x^2)}{2\sqrt{x}(3+x^2)^2}$$

Curve Sketching:

1. Plot all points: x-int, y-int, max, min, I.P.,
2. Plot all asymptotes (Check behaviour near VA.)
3. Use intervals of inc/dec and concavity to connect everything

Sketch $f(x) = x^3 - 12x^2 + 36x$

$f'(x) = 3x^2 - 24x + 36$

$f''(x) = 6x - 24$

① x-int

$0 = x^3 - 12x^2 + 36x$

$0 = x(x^2 - 12x + 36)$

$0 = x(x-6)(x-6)$

$x = 0 \mid x - 6 = 0$, $x = 6$

$(0,0)$ $(6,0)$

② y-int $(x=0)$

$y = (0)^3 - 12(0)^2 + 36(0)$

$y = 0$

$(0,0)$

③ Intervals of Inc/Dec

$f'(x) = 3x^2 - 24x + 36$

$f'(x) = 3(x^2 - 8x + 12)$

$f'(x) = 3(x-6)(x-2)$

max min

$+ \quad - \quad +$

(1) 2 (3) 6 (7)

$0 = 3(x-6)(x-2)$

CV: $x = 6$, $x = 2$

Increasing on $(-\infty, 2) + (6, \infty)$

Decreasing on $(2, 6)$

④ max/min:

max @ $x = 2$

$f(x) = x^3 - 12x^2 + 36x$

$f(2) = (2)^3 - 12(2)^2 + 36(2)$

$f(2) = 8 - 48 + 72$

$f(2) = 32$

$(2, 32)$

min @ $x = 6$

$f(x) = x^3 - 12x^2 + 36x$

$f(6) = (6)^3 - 12(6)^2 + 36(6)$

$f(6) = 216 - 432 + 216$

$f(6) = 0$

$(6, 0)$

⑤ Intervals of Concavity:

$f''(x) = 6x - 24$

$f''(x) = 6(x-4)$

$0 = 6(x-4)$

CV: $x = 4$

IP

$- \quad +$

(0) 4 (5)

CD on $(-\infty, 4)$

CU on $(4, \infty)$

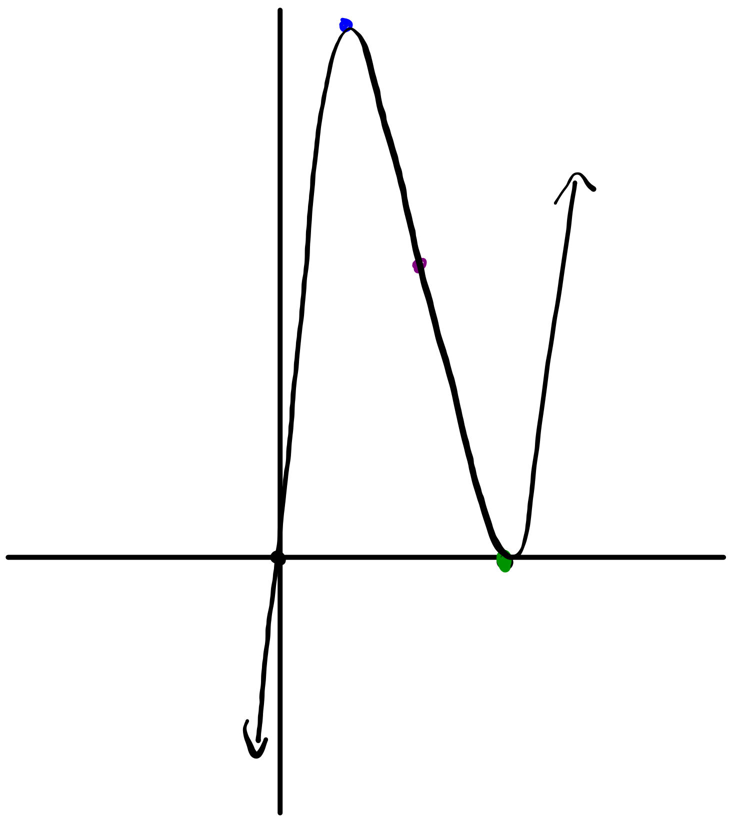
⑥ Inflection Point @ $x = 4$

$f(x) = x^3 - 12x^2 + 36x$

$f(4) = (4)^3 - 12(4)^2 + 36(4)$

$f(4) = 64 - 192 + 144$

$f(4) = 16$



Curve Sketching

Example:

Examine the function $f(x) = 3x^5 - 5x^3$ with respect to…

- Intercepts $f(x)$
- ~~Symmetry~~
- Asymptotes (No asymptotes for polynomial functions)
- Intervals of Increase or Decrease $f'(x)$
- Local Maximum and Minimum values $f(x)$
- $f''(x)$ Concavity and Points of Inflection $f(x)$
- Sketch the Curve

$f(x) = 3x^5 - 5x^3$ $\quad f'(x) = 15x^4 - 15x^2$ $\quad f''(x) = 60x^3 - 30x$

$f(x) = x^3(3x^2 - 5)$ $\quad f'(x) = 15x^2(x^2 - 1)$ $\quad f''(x) = 30x(2x^2 - 1)$

$f'(x) = 15x^2(x-1)(x+1)$

① x-int (y=0)

$f(x) = x^3(3x^2-5)$

$0 = x^3(3x^2-5)$

$x^3 = 0$ | $3x^2 - 5 = 0$

$x = 0$ | $\frac{3x^2}{3} = \frac{5}{3}$

(0,0) | $x^2 = \frac{5}{3}$

$x = \pm\sqrt{5/3}$

(1.29, 0) + (-1.29, 0)

② y-int (x=0)

$f(x) = 3x^5 - 5x^3$

$f(0) = 3(0)^5 - 5(0)^3$

$f(0) = 0$

(0,0)

③ Intervals of Inc/Dec.

$f'(x) = 15x^2(x-1)(x+1)$

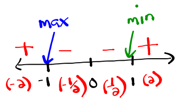
$0 = 15x^2(x-1)(x+1)$

$15x^2 = 0$ | $x - 1 = 0$ | $x + 1 = 0$

$x^2 = 0$ | $x = 1$ | $x = -1$

$x = 0$

CV: $x = -1, 0, 1$



Increasing on $(-\infty, -1) + (1, \infty)$

Decreasing on $(-1, 0) + (0, 1)$ or $(-1, 1)$

④ max @ $\underline{x = -1}$

$f(x) = 3x^5 - 5x^3$

$f(-1) = 3(-1)^5 - 5(-1)^3$

$f(-1) = -3 + 5$

$f(-1) = 2$

(-1, 2)

⑤ min @ $\underline{x = 1}$

$f(x) = 3x^5 - 5x^3$

$f(1) = 3(1)^5 - 5(1)^3$

$f(1) = 3 - 5$

$f(1) = -2$

(1, -2)

⑥ Intervals of Concavity:

$f''(x) = 30x(2x^2 - 1)$

$0 = 30x(2x^2 - 1)$

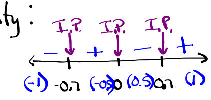
$30x = 0$ | $2x^2 - 1 = 0$

$x = 0$ | $2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{1/2}$

$x \simeq \pm 0.707$



CD on $(-\infty, -0.7) + (0, 0.7)$

CU on $(-0.7, 0) + (0.7, \infty)$

CV: $x = -0.7, 0, 0.7$

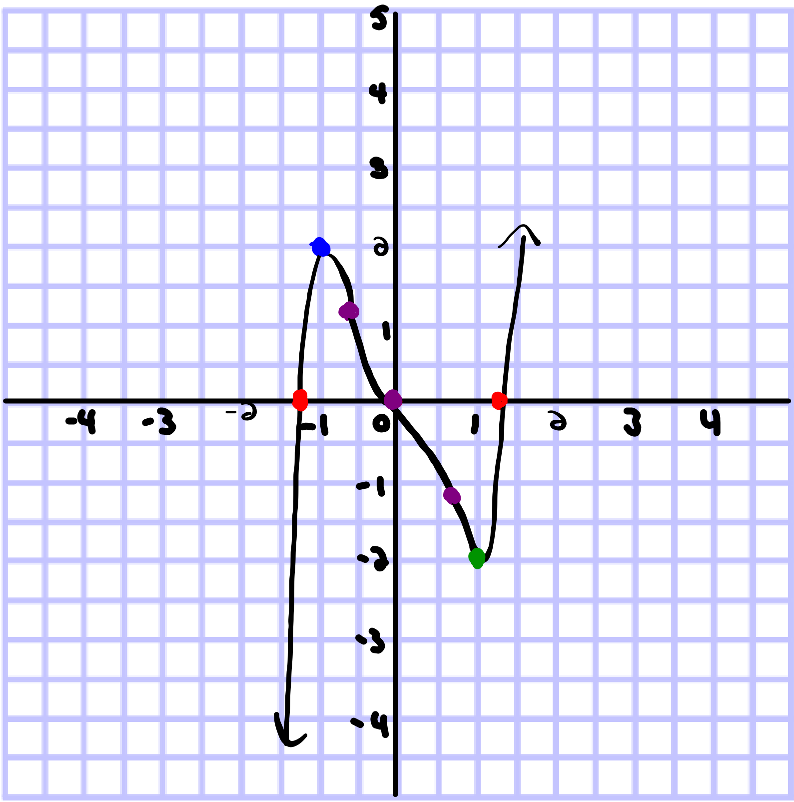
⑦ Inflection Points

$f(x) = 3x^5 - 5x^3$

$f(-0.7) = 3(-0.7)^5 - 5(-0.7)^3 = -0.504 + 1.715 = 1.2$ $\quad (-0.7, 1.2)$

$f(0) = 3(0)^5 - 5(0)^3 = 0 - 0 = 0$ $\quad (0,0)$

$f(0.7) = 3(0.7)^5 - 5(0.7)^3 = 0.504 - 1.715 = -1.2$ $\quad (0.7, -1.2)$



5
4
3
2
1
-4
-3
-2
-1
0
1
2
3
4
-1
-2
-3
-4

Curve Sketching

Examine the function $f(x) = \frac{x^2}{1-x^2}$ with respect to... $f'(x) = \frac{2x}{(1-x^2)^2}$

$f''(x) = \frac{2(1+3x^2)}{(1-x^2)^3}$

- Intercepts $f(x)$
- ~~Symmetry~~
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

① x-int $(y=0)$

$f(x) = \frac{x^2}{1-x^2}$

$0 = \frac{x^2}{1-x^2}$

$0 = x^2$

$0 = x$

$(0,0)$

② y-int $(x=0)$

$f(x) = \frac{x^2}{1-x^2}$

$f(0) = \frac{(0)^2}{1-(0)^2}$

$f(0) = \frac{0}{1} = 0$

$(0,0)$

③ Vertical Asymptote: (zeroes of the denominator)

$f(x) = \frac{x^2}{1-x^2}$

VA: $1-x^2=0$

$(1-x)(1+x)=0$

$1-x=0 \quad | \quad 1+x=0$

$\boxed{1=x} \quad \boxed{x=-1}$

④ Horizontal Asymptote:

$\lim_{x\to\infty} \frac{x^2}{1-x^2} = \frac{1}{-1} = -1$

$\boxed{y=-1}$

⑤ Intervals of Inc/Dec.

$f'(x) = \frac{2x}{(1-x^2)^2}$

$2x=0 \quad | \quad (1-x^2)^2=0$

$x=0 \quad | \quad 1-x^2=0$

$1=x^2$

$\pm 1 = x$

CV: $x=-1,0,1$

min

$- \quad - \quad + \quad +$

$(-2) \quad -1 \quad (-0.5) \quad 0 \quad (0.5) \quad 1 \quad (2)$

Increasing on $(0,\infty)$

Decreasing on $(-\infty,0)$

⑥ min @ $\underline{x=0}$

$f(x) = \frac{x^2}{1-x^2}$

$f(0) = \frac{(0)^2}{1-(0)^2} = 0$

$(0,0)$

⑦ Intervals of Concavity

$f''(x) = \frac{2(1+3x^2)}{(1-x^2)^3}$

I.P. I.P.

$- \quad + \quad -$

$(-2) \quad -1 \quad (0) \quad 1 \quad (2)$

$2(1+3x^2)=0 \quad | \quad (1-x^2)^3=0$

$1+3x^2=0 \quad | \quad 1-x^2=0$

$3x^2=-1 \quad | \quad 1=x^2$

$x^2=-\frac{1}{3} \quad | \quad \pm 1 = x$

Not Possible

(Numerator is always positive)

CD on $(-\infty,-1)$ + $(1,\infty)$

CU on $(-1,1)$

⑧ Inflection Points:

when $\underline{x=-1}$

$f(x) = \frac{x^2}{1-x^2}$

$f(-1) = \frac{(-1)^2}{1-(-1)^2} = \frac{1}{0} =$ und.

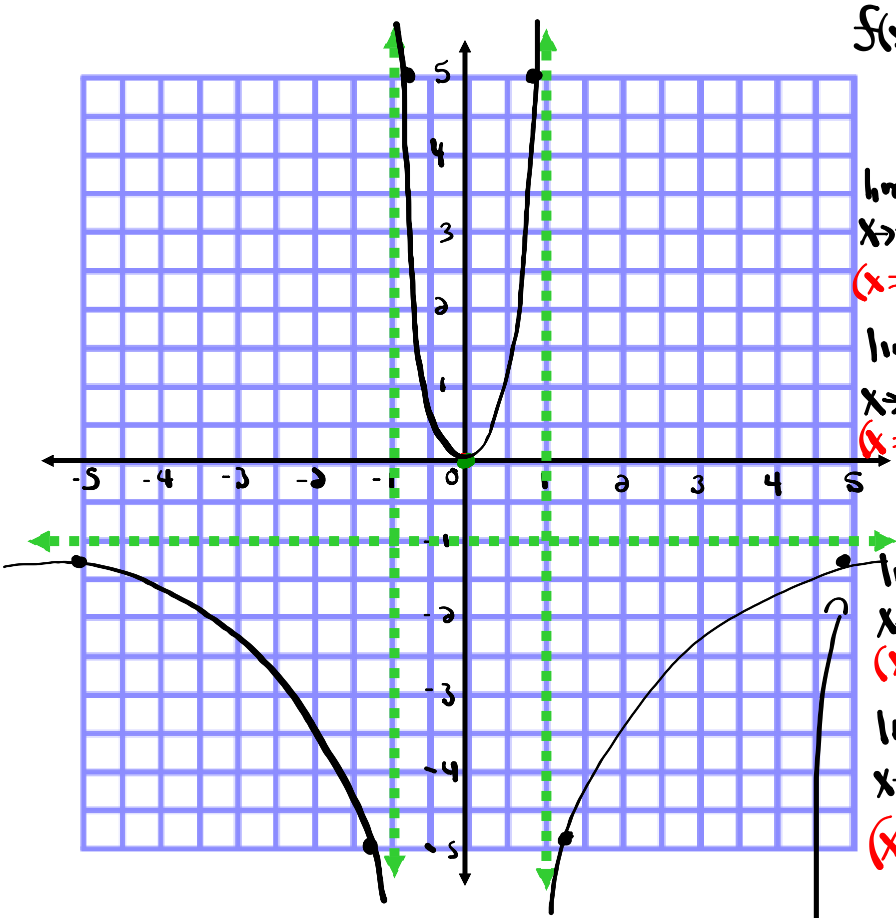
when $\underline{x=1}$

$f(x) = \frac{x^2}{1-x^2}$

$f(1) = \frac{(1)^2}{\;} = \frac{1}{\;} =$ u

$f(x) = \frac{x^2}{1-x^2}$

$f(x) = \frac{x^2}{(1-x)(1+x)}$



$\lim_{x \to -1^-} f(x) = \frac{(+)}{(+)(-)} = -\infty$ (x=-1.1)

$\lim_{x \to -1^+} f(x) = \frac{(+)}{(+)(+)} = +\infty$ (x=-0.9)

$\lim_{x \to 1^-} f(x) = \frac{(+)}{(+)(+)} = +\infty$ (x=0.9)

$\lim_{x \to 1^+} f(x) = \frac{(+)}{(-)(+)} = -\infty$ (x=1.1)

$f(5) = \frac{(5)^2}{1-(5)^2} = \frac{25}{-24}$

Synthetic Sub:

$-27+45+6-24$

$$x^3+5x^2-2x-24$$

Find an x value that makes the polynomial equal 0

$(2)^3+5(2)^2-2(2)-24$

$8+20-4-24$

0

2	1	5	-2	-24
		2	14	24
x=2	1	7	12	0

$(x-2)(x^2+7x+12)$ ← simple trinomial

$3+4=7$

$3\times 4=12$

$(x-2)(x+3)(x+4)$

Using Factor Theorem

$x^3 + 5x^2 - 2x - 24$

$x = -3$

$(-3)^3 + 5(-3)^2 - 2(-3) - 24$

$\boxed{x+3 \text{ is a factor}}$

$-27 + 45 + 6 - 24$

0

$$
\begin{array}{r|l}
 & x^2 + 2x - 8 \\
x+3 & x^3 + 5x^2 - 2x - 24 \\
 & -(x^3 + 3x^2) \\
 & 2x^2 - 2x - 24 \\
 & -(2x^2 + 6x) \\
 & -8x - 24 \\
 & -(-8x - 24) \\
 & 0
\end{array}
$$

$(x+3)(x^2+2x-8)$

$(x+3)(x+4)(x-2)$

Curve Sketching:

$$f(x) = \frac{x^2+5x+6}{x^2-9} = \frac{(x+2)\cancel{(x+3)}}{(x-3)\cancel{(x+3)}} = \frac{x+2}{x-3}$$

x int:

$x+2=0$

$x=-2$

$(-2,0)$

y int:

$f(0) = \frac{0+2}{0-3}$

$f(0) = -\frac{2}{3}$

$(0,-2/3)$

VA:

$x-3=0$

$x=3$

HA: $\lim_{x\to\infty} \frac{x+2}{x-3} = 1$

$y=1$

Hole:

$x+3=0$

$x=-3$

$f(-3) = \frac{-3+2}{-3-3}$

$f(-3) = \frac{1}{6}$