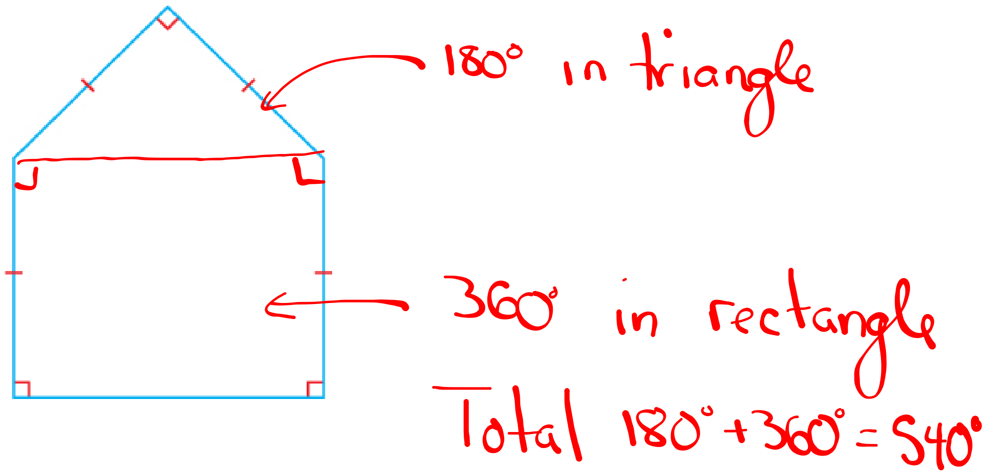
2.4 Angle Properties in Polygons

GOAL

Determine properties of angles in polygons, and use these properties to solve problems.

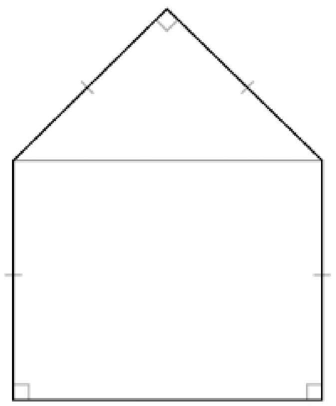
EXPLORE...

- A pentagon has three right angles and four sides of equal length, as shown. What is the sum of the measures of the angles in the pentagon?



SAMPLE ANSWER

I drew a diagonal joining the two angles that are not right angles. This cut the pentagon into a rectangle and a triangle. I knew that the quadrilateral was a rectangle, not a trapezoid, because the two right angles share an arm, so their other arms must be parallel. As well, the other arms are equal length. I knew that the sum of the measures of the angles in a rectangle is 360° and the sum of the measures of the angles in a triangle is 180°, so the sum of the measures of the angles in the pentagon must be 540°.

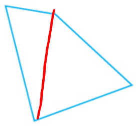


INVESTIGATE the Math

In Lesson 2.3, you proved properties involving the interior and exterior angles of triangles. You can use these properties to develop general relationships involving the interior and exterior angles of polygons.

? How is the number of sides in a polygon related to the sum of its interior angles and the sum of its exterior angles? $180°(n-2)$

Part 1 Interior Angles

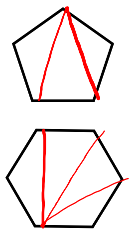


A. Giuseppe says that he can determine the sum of the measures of the interior angles of this quadrilateral by including the diagonals in the diagram. Is he correct? Explain.

B. Determine the sum of the measures of the interior angles of any quadrilateral.

C. Draw the polygons listed in the table below. Create triangles to help you determine the sum of the measures of their interior angles. Record your results in a table like the one below.

Polygon	Number of Sides	Number of Triangles	Sum of Angle Measures
triangle	3	1	180°
quadrilateral	4	2	360°
pentagon	5	3	540°
hexagon	6	4	720°
heptagon	7	5	900°
octagon	8	6	1080°

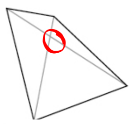


D. Make a conjecture about the relationship between the sum of the measures of the interior angles of a polygon, S, and the number of sides of the polygon, n.

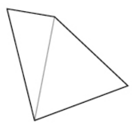
E. Use your conjecture to predict the sum of the measures of the interior angles of a dodecagon (12 sides). Verify your prediction using triangles.

Answers

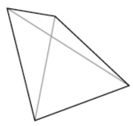
A. Approach 1: Giuseppe is not correct. I tried his strategy. The diagonals cut the quadrilateral into four triangles. The sum of the measures of the angles in four triangles is 4(180°) or 720°. The sum of the measures of the angles in a quadrilateral is actuallv 360°.



Approach 2: Giuseppe is correct if he considers only one diagonal in the quadrilateral. One diagonal separates the quadrilateral into two triangles. The angles of the triangles form the angles of the quadrilateral. This means that the sum of the measures of the angles in any quadrilateral is 360°.



Approach 3: Yes, Giuseppe is correct. I drew the diagonals and noticed that they cut the quadrilateral into four triangles. The sum of the measures of the angles in the four triangles is 4(180°) or 720°. But one angle in each triangle occurs where the two diagonals intersect. The sum of the measures of these angles is 360°. I subtracted 360° from 720°, giving the correct sum of the measures of the angles in the quadrilateral.



B. The sum of the measures of the angles in a quadrilateral is 360°.

C. For example:

Polygon	Number of Sides	Number of Triangles	Sum of Angle Measures
triangle	3	1	180°
quadrilateral	4	2	360°
pentagon	5	3	540°
hexagon	6	4	720°
heptagon	7	5	900°
octagon	8	6	1080°

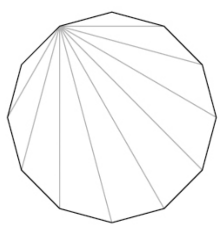
D. The number of triangles in a polygon is two less than the number of sides. To determine the sum of the measures of the angles in any polygon, subtract 2 from the number of sides and then multiply by 180°. This is my conjecture: The sum of the measures of the interior angles in a polygon, $S(n)$, is:

$S(n) = 180°(n - 2)$

E. For example, I predict that the sum of the measures of the angles in a dodecagon is

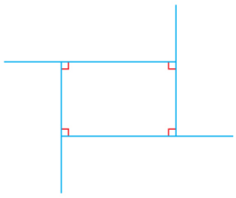
$$S(n) = 180°(n - 2)$$
$$S(12) = 180°(12 - 2)$$
$$S(12) = 1800°$$

I drew a dodecagon. Then I drew all the diagonals from one of the vertices. There are 10 triangles in my diagram, so the sum of the measures of the angles in a dodecagon is 10(180°) or 1800°.



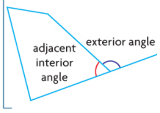
Part 2 Exterior Angles

F. Draw a rectangle. Extend each side of the rectangle so that the rectangle has one exterior angle for each interior angle. Determine the sum of the measures of the exterior angles.

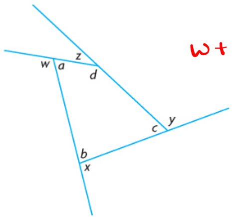


Communication Tip

When a side of a polygon is extended, two angles are created. The angle that is considered to be the exterior angle is adjacent to the interior angle at the vertex.



G. What do you notice about the sum of the measures of each exterior angle of your rectangle and its adjacent interior angle? Would this relationship also hold for the exterior and interior angles of the irregular quadrilateral shown? Explain.



$w + x + y + z = 360°$

H. Make a conjecture about the sum of the measures of the exterior angles of any quadrilateral. Test your conjecture.

I. Draw a pentagon. Extend each side of the pentagon so that the pentagon has one exterior angle for each interior angle. Based on your diagram, revise your conjecture to include pentagons. Test your revised conjecture.

J. Do you think your revised conjecture will hold for polygons that have more than five sides? Explain and verify by testing.

Answers

F. For example, each exterior angle has a measure of 90°. The sum of the measures is 360°.

G. They are equal. This would not hold for an irregular quadrilateral. Alternatively, each exterior angle is supplementary to its adjacent interior angle. This would be true for any quadrilateral, because this pair of angles forms a straight line.

H. My conjecture: The sum of the measures of the exterior angles of any quadrilateral is 360°.

To test my conjecture, I wrote each exterior angle as the supplement of its adjacent interior angle a, b, c, and d.

$w = (180° - a)$, $x = (180° - b)$, $y = (180° - c)$, and $z = (180° - d)$

The sum of the measures of the exterior angles is

$S = w + x + y + z$

$S = (180° - a) + (180° - b) + (180° - c) + (180° - d)$ Substitution

$S = 720° - (a + b + c + d)$

$a + b + c + d = 360°$ Sum of the measures of the interior angles of a quadrilateral

$S = 720° - (360°)$ Substitution

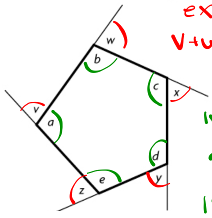
$S = 360°$

I. My revised conjecture: The sum of the measures of the exterior angles of a pentagon will also be 360°.

I drew a pentagon with interior angle measures a, b, c, d, and e, and exterior angle measures v, w, x, y, and z, as shown.

I wrote each exterior angle as the supplement of its adjacent interior angle.

$v = (180° - a)$, $w = (180° - b)$, $x = (180° - c)$, $y = (180° - d)$, and $z = (180° - e)$



exterior angles: $v + w + x + y + z = 360°$

interior angles: pentagon: $n = 5$

$180°(5-2)$

$180°(3)$

$540°$

The sum of the measures of the exterior angles is

$S = v + w + x + y + z$

$S = (180° - a) + (180° - b) + (180° - c) + (180° - d) + (180° - e)$ Substitution

$S = 900° - (a + b + c + d + e)$

$a + b + c + d + e = 540°$ Sum of the measures of the interior angles of a pentagon

$S = 900° - 540°$ Substitution

$S = 360°$

My conjecture is correct.

what is the measure of angle "a"

$\frac{180°(n-2)}{n}$

$\frac{180°(5-2)}{5}$

$\frac{540°}{5}$

$108°$

J. Yes, it will always hold. You can think of the sum of the measures of the exterior angles as the angles you turn when walking around the shape. Since a complete turn of a circle is 360°, I think the sum of the measures of the angles will also be 360°.

angle a, b, c, d, e are all equal to 108°

Reflecting

K. Compare your results for the sums of the measures of the interior angles of polygons with your classmates' results. Do you think your conjecture from part *D* will be true for any polygon? Explain.

L. Compare your results for the sums of the measures of the exterior angles of polygons with your classmates' results. Do you think your conjecture from part I will apply to any polygon? Explain.

Answers

K. Yes, I do. A polygon with n sides has $(n - 3)$ diagonals that can be drawn from one vertex, cutting the polygon into $(n - 2)$ triangles. Therefore, my conjecture should be valid for all polygons.

L. Yes, I do. The sum of the measures of the exterior angles is $360°$ for all the polygons we tried. It is reasonable to think that my conjecture will be valid for all polygons.

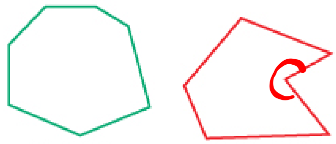
APPLY the Math

EXAMPLE 1 Reasoning about the sum of the interior angles of a polygon

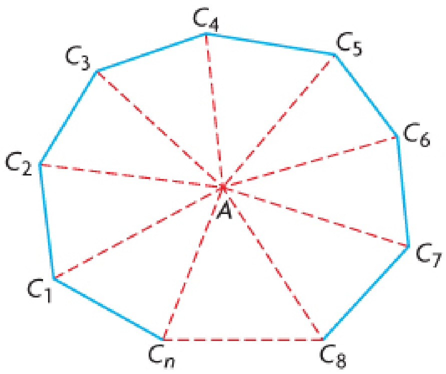
Prove that the sum of the measures of the interior angles of any n-sided **convex polygon** can be expressed as $180°(n - 2)$.

convex polygon

A polygon in which each interior angle measures less than 180°.



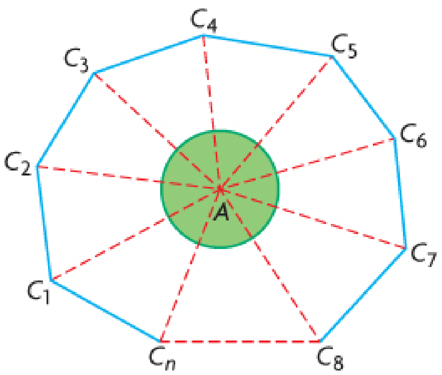
Viktor's Solution



I drew an n-sided polygon. I represented the nth side using a broken line. I selected a point in the interior of the polygon and then drew line segments from this point to each vertex of the polygon. The polygon is now separated into n triangles.

The sum of the measures of the angles in each triangle is 180°.

The sum of the measures of the angles in n triangles is $n(180°)$.



Two angles in each triangle combine with angles in the adjacent triangles to form two interior angles of the polygon.

Each triangle also has an angle at vertex A. The sum of the measures of the angles at A is 360° because these angles make up a complete rotation. These angles do not contribute to the sum of the interior angles of the polygon.

The sum of the measures of the interior angles of the polygon, $S(n)$, where n is the number of sides of the polygon, can be expressed as:

$S(n) = 180°n - 360°$

$S(n) = 180°(n - 2)$

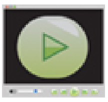
The sum of the measures of the interior angles of a convex polygon can be expressed as $180°(n - 2)$.

EXAMPLE 1 Reasoning about the sum of the interior angles of a polygon

Prove that the sum of the measures of the interior angles of any n-sided **convex polygon** can be expressed as $180°(n - 2)$.

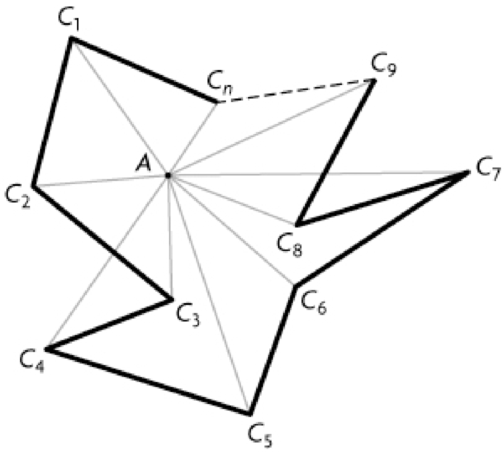
Your Turn

Explain why Viktor's solution cannot be used to show whether the expression $180°(n - 2)$ applies to non-convex polygons.



Answer

Viktor's strategy only works if the triangles formed by the line segments from a point on the interior of the polygon to each vertex of the polygon lie entirely inside the polygon. To illustrate this, I drew a non-convex polygon and tried Viktor's strategy. I noticed that line segments AC_4 and AC_7 each formed a triangle that had whole and partial angles outside the polygon.



EXAMPLE 2 Reasoning about angles in a regular polygon

Outdoor furniture and structures like gazebos sometimes use a regular hexagon in their building plan. Determine the measure of each interior angle of a regular hexagon.

Nazra's Solution

Let $S(n)$ represent the sum of the measures of the interior angles of the polygon, where n is the number of sides of the polygon.

$$S(n) = 180°(n - 2)$$

A hexagon has six sides, so $n = 6$.

$$S(6) = 180°[(6) - 2]$$
$$S(6) = 720°$$
$$\frac{720°}{6} = 120°$$

Since the measures of the angles in a regular hexagon are equal, each angle must measure $\frac{1}{6}$ of the sum of the angles.

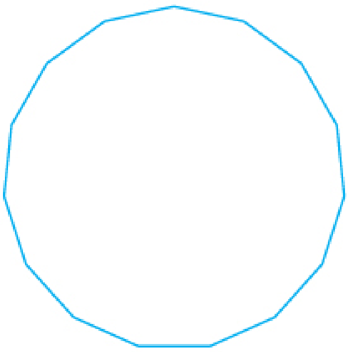
The measure of each interior angle of a regular hexagon is 120°.

EXAMPLE 2 Reasoning about angles in a regular polygon

Outdoor furniture and structures like gazebos sometimes use a regular hexagon in their building plan. Determine the measure of each interior angle of a regular hexagon.

Your Turn

Determine the measure of each interior angle of a regular 15-sided polygon (a pentadecagon).



Answer

The sum of the measures of the angles in a pentadecagon is $13(180°)$ or $2340°$. For a regular pentadecagon,

each angle is $\frac{2340}{15}$ or $156°$.

EXAMPLE 3 Visualizing tessellations

A floor tiler designs custom floors using tiles in the shape of regular polygons. Can the tiler use congruent regular octagons and congruent squares to tile a floor, if they have the same side length?

Vanessa's Solution

$S(n) = 180°(n - 2)$

$S(8) = 180°[(8) - 2]$

$S(8) = 1080°$

$\frac{1080°}{8} = 135°$

The measure of each interior angle in a regular octagon is 135°.

The measure of each internal angle in a square is 90°.

Since an octagon has eight sides, $n = 8$.

First, I determined the sum of the measures of the interior angles of an octagon. Then I determined the measure of each interior angle in a regular octagon.

Two octagons fit together, forming an angle that measures:

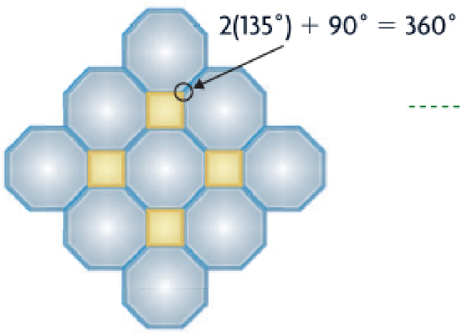
$2(135°) = 270°$.

This leaves a gap of 90°.

$2(135°) + 90° = 360°$

A square can fit in this gap if the sides of the square are the same length as the sides of the octagon.

I knew that three octagons would not fit together, as the sum of the angles would be greater than 360°.



I drew what I had visualized using dynamic geometry software.

The tiler can tile a floor using regular octagons and squares when the polygons have the same side length.

In Summary

Key Idea

- You can prove properties of angles in polygons using other angle properties that have already been proved.

Need to Know

- The sum of the measures of the interior angles of a convex polygon with n sides can be expressed as $180°(n - 2)$.
- The measure of each interior angle of a regular polygon is $\frac{180°(n-2)}{n}$.
- The sum of the measures of the exterior angles of any convex polygon is 360°. (does not matter how many sides)

n = # sides

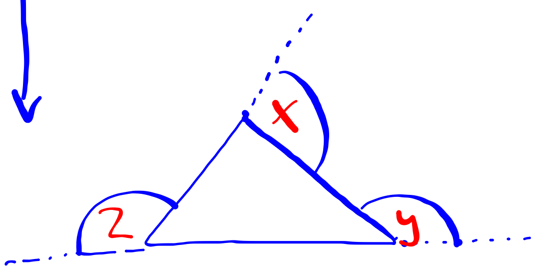
Ex:

triangle: $n=3$

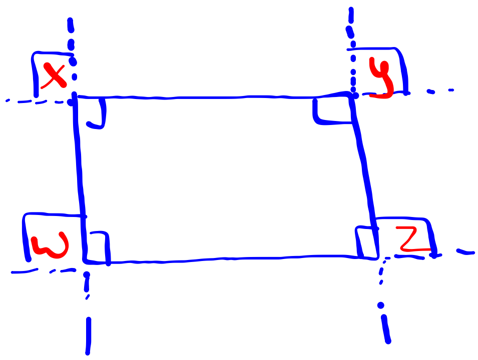
hexagon: $n=6$

Assignment: **pgs. 99 - 101**

1, 2, 3, 6, 7, 8, 10a, 11, 13



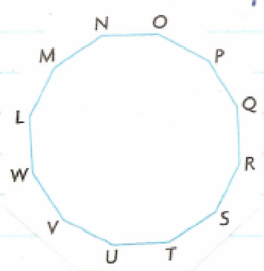
$x+y+z=360°$



$w+x+y+z=360°$

SOLUTIONS ⇒ 2.4 Angle Properties in Polygons

1.a) Determine the sum of the measures of the interior angles of a regular dodecagon.



$S = 180°(n-2)$
$S = 180°(12-2)$
$S = 180°(10)$
$S = 1800°$

b) Determine the measure of each interior angle of a regular dodecagon.

$$\text{Measure of each interior angle} = \frac{180°(n-2)}{n}$$
$$= \frac{180°(12-2)}{12}$$
$$= \frac{180°(10)}{12}$$
$$= \frac{1800°}{12}$$
$$= 150°$$

2. Determine the sum of the measures of the angles in a 20-sided convex polygon.

$S = 180°(n-2)$
$S = 180°(20-2)$
$S = 180°(18)$
$S = 3240°$

3. The sum of the measures of the interior angles of an unknown polygon is 3060°. Determine the number of sides that the polygon has.

$$S = 180°(n-2)$$
$$3060° = 180°n - 360°$$
$$3060° + 360° = 180°n$$
$$\frac{3420°}{180°} = \frac{180°n}{180°}$$
$$19 = n$$

6. Determine the measure of each interior angle of a loonie.

$$\text{Measure of each interior angle} = \frac{180°(n-2)}{n}$$
$$= \frac{180°(11-2)}{11}$$
$$= \frac{180°(9)}{11}$$
$$\approx 147°$$

Approximately equal to

7. Each interior angle of a regular convex polygon measures 140°.

a) Prove that the polygon has nine sides.

Measure of each interior angle $= \frac{180°(n-2)}{n}$

$$140° = \frac{180°(n-2)}{n}$$

$$140°n = 180°n - 360°$$

$$140°n - 180°n = -360°$$

$$\frac{-40°n}{-40°} = \frac{-360°}{-40°}$$

$$n = 9$$

b) Verify that the sum of the measures of the exterior angles is 360°

$180° - 140° = 40°$ (Each exterior angle)

$9(40°) = 360°$

8a) Determine the measure of each exterior angle of a regular octagon.

Each exterior angle $= \frac{360°}{8}$

$= 45°$

b) Use your answer for part a) to determine the measure of each interior angle of a regular octagon.

Each interior angle $= 180° - 45°$ {Supplementary Angles}

$= 135°$

c) Use your answer for part b) to determine the sum of the interior angles of a regular octagon.

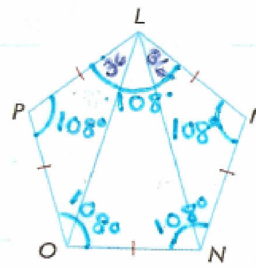
Sum of interior angles $= 8(135°)$
$= 1080°$

d) Use the function $S(n) = 180°(n-2)$ to determine the sum of the interior angles of a regular octagon. Compare your answer with the sum you determined in part (c).

$S = 180°(n-2)$
$S = 180°(8-2)$
$S = 180°(6)$
$S = 1080°$

* The answers for part (c) and part (d) are the same.

10. LMNOP is a regular pentagon.



a) Determine the measure of $\angle OLN$.

Each interior angle of the pentagon $= \frac{180°(n-2)}{n}$

$= \frac{180°(5-2)}{5}$

$= \frac{180°(3)}{5}$

$= 108°$

$\angle PLO$ and $\angle MLN = \frac{180° - 108°}{2}$

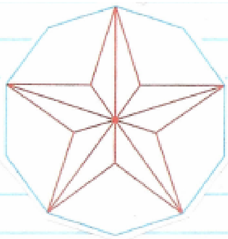
$= \frac{72°}{2}$

$= 36°$ (Isosceles Triangles)

$\angle OLN = 108° - 36° - 36°$

$= 36°$

11. Sandy designed this logo for the jerseys worn by her softball team. She told the graphic artist that each interior angle of the regular decagon should measure 162°, based on this calculation:



$$S = \frac{180°(10-1)}{10}$$

$$S = \frac{1620°}{10}$$

$$S = 162°$$

Identify the error she made and determine the correct angle.

* The error occurs in the first line.
The correct formula is: $S(n) = \frac{180°(n-2)}{n}$

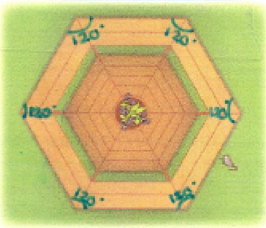
$$\Rightarrow S(10) = \frac{180°(10-2)}{10}$$

$$= \frac{180°(8)}{10}$$

$$= 144°$$

13. Martin is planning to build a hexagonal picnic table, as shown.

a) Determine the angles at the ends of each piece of wood that Martin needs to cut for the seats.



Each interior angle of a hexagon $= \frac{180°(n-2)}{n}$

$= \frac{180°(6-2)}{6}$

$= \frac{180°(4)}{6}$

$= 120°$

Angles at the end of each piece of wood for the seat

$\hookrightarrow \frac{120°}{2} = 60°$

b) How would these angles change if Martin decided to make an octagonal table instead?

Each interior angle of an octagon $= \frac{180°(n-2)}{n}$

$= \frac{180°(8-2)}{8}$

$= \frac{180°(6)}{8}$

$= 135°$

Angles at the end of each piece of wood for the seat

$\hookrightarrow \frac{135°}{2} = 67.5°$

Attachments

PM11-2s4-interior.gsp

PM11-2s4-exterior.gsp

2s4e1 finalt.mp4

2s4e2 finalt.mp4

2s4e3 finalt.mp4