

Making a Complete Sketch

- ① Plot all points: x -int, y -int, max, min, I.P.,
- ② Plot all asymptotes (Check behaviour near VA.)
- ③ use intervals of inc/dec and concavity to connect everything

Example:

Examine the function $f(x) = 3x^5 - 5x^3$ with respect to...

- Intercepts $f(x)$
- ~~Symmetry~~
- Asymptotes (No asymptotes for polynomial functions)
- Intervals of Increase or Decrease $f'(x)$
- Local Maximum and Minimum values $f'(x)$
- $f''(x)$ Concavity and Points of Inflection $f''(x)$
- Sketch the Curve

$$f(x) = 3x^5 - 5x^3 \quad f'(x) = 15x^4 - 15x^2 \quad f''(x) = 60x^3 - 30x$$

$$f(x) = x^3(3x^2 - 5) \quad f'(x) = 15x^2(x^2 - 1) \quad f''(x) = 30x(x^2 - 1)$$

$$f'(x) = 15x^2(x-1)(x+1)$$

① x-int ($y=0$)

$$f(x) = x^3(3x^2 - 5)$$

$$0 = x^3(3x^2 - 5)$$

$$x^3 = 0 \quad 3x^2 - 5 = 0$$

$$x = 0 \quad 3x^2 = 5$$

$$(0,0) \quad x^2 = \frac{5}{3}$$

$$x = \pm\sqrt{\frac{5}{3}}$$

$$(1.29, 0) \quad + (-1.29, 0)$$

② y-int ($x=0$)

$$f(x) = 3x^5 - 5x^3$$

$$f(0) = 3(0)^5 - 5(0)^3$$

$$f(0) = 0$$

$$(0,0)$$

③ Intervals of Inc/Dec.

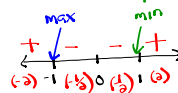
$$f'(x) = 15x^2(x-1)(x+1)$$

$$0 = 15x^2(x-1)(x+1)$$

$$15x^2 = 0 \quad x-1 = 0 \quad x+1 = 0$$

$$x^2 = 0 \quad x = 1 \quad x = -1$$

$$x = 0$$



Increasing on $(-\infty, -1) + (1, \infty)$
 Decreasing on $(-1, 0) + (0, 1)$
 or $(-1, 1)$

CV: $x = -1, 0, 1$

④ max @ $x = -1$

$$f(x) = 3x^5 - 5x^3$$

$$f(-1) = 3(-1)^5 - 5(-1)^3$$

$$f(-1) = -3 + 5$$

$$f(-1) = 2$$

$$(-1, 2)$$

⑤ min @ $x = 1$

$$f(x) = 3x^5 - 5x^3$$

$$f(1) = 3(1)^5 - 5(1)^3$$

$$f(1) = 3 - 5$$

$$f(1) = -2$$

$$(1, -2)$$

⑥ Intervals of Concavity:

$$f''(x) = 30x(x^2 - 1)$$

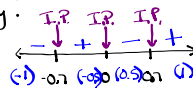
$$0 = 30x(x^2 - 1)$$

$$30x = 0 \quad x^2 - 1 = 0$$

$$x = 0 \quad x^2 = 1$$

$$x = \pm\sqrt{1}$$

$$x = \pm 1$$



CD on $(-\infty, -1) + (0, 1)$
 CU on $(-1, 0) + (1, \infty)$

CV: $x = -0.7, 0, 0.7$

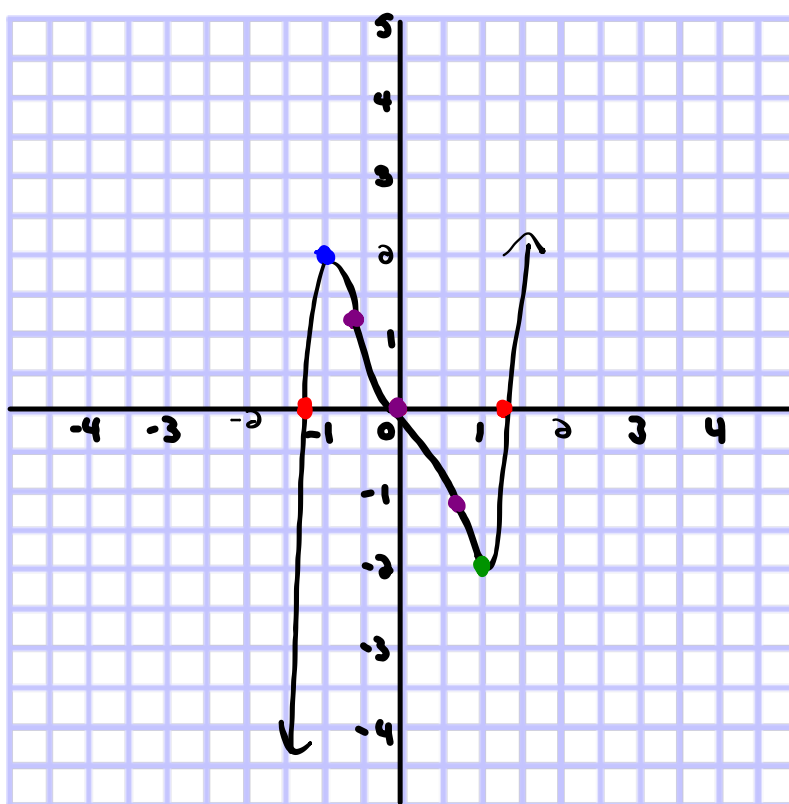
⑦ Inflection Points

$$f(x) = 3x^5 - 5x^3$$

$$f(-0.7) = 3(-0.7)^5 - 5(-0.7)^3 = -0.504 + 1.715 = 1.2 \quad (-0.7, 1.2)$$

$$f(0) = 3(0)^5 - 5(0)^3 = 0 - 0 = 0 \quad (0, 0)$$

$$f(0.7) = 3(0.7)^5 - 5(0.7)^3 = 0.504 - 1.715 = -1.2 \quad (0.7, -1.2)$$



Assignment:

$$f(x) = x^2 + x^3 \quad f'(x) = 2x + 3x^2 \quad f''(x) = 2 + 6x$$

$$f(x) = x^2(1+x) \quad f'(x) = x(2+3x) \quad f''(x) = 2(1+3x)$$

① x-int ($y=0$) ② y-int ($x=0$)

$$f(x) = x^2(1+x)$$

$$0 = x^2(1+x)$$

$$x^2 = 0 \quad | \quad 1+x = 0$$

$$x = 0 \quad | \quad x = -1$$

$$(0,0) \quad | \quad (-1,0)$$

$$f(x) = x^2 + x^3$$

$$f(0) = (0)^2 + (0)^3$$

$$f(0) = 0 + 0$$

$$f(0) = 0$$

$$(0,0)$$

③ Intervals of Inc/Dec.

$$f'(x) = x(2+3x)$$

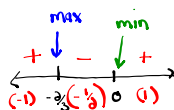
$$0 = x(2+3x)$$

$$x = 0 \quad | \quad 2+3x = 0$$

$$3x = -2$$

$$x = -\frac{2}{3}$$

$$CV: x = -\frac{2}{3}, 0$$

Increasing on $(-\infty, -\frac{2}{3}) \cup (0, \infty)$ Decreasing on $(-\frac{2}{3}, 0)$ ④ max @ $x = -\frac{2}{3}$

$$f(x) = x^2 + x^3$$

$$f(-\frac{2}{3}) = (-\frac{2}{3})^2 + (-\frac{2}{3})^3$$

$$f(-\frac{2}{3}) = \frac{4}{9} - \frac{8}{27}$$

$$f(-\frac{2}{3}) = \frac{12}{27} - \frac{8}{27} = \frac{4}{27}$$

$$(-\frac{2}{3}, \frac{4}{27}) \text{ or } (-0.6, 0.15)$$

⑤ min @ $x = 0$

$$f(x) = x^2 + x^3$$

$$f(0) = (0)^2 + (0)^3$$

$$f(0) = 0 + 0 = 0$$

$$(0,0)$$

⑥ Intervals of Concavity:

$$f''(x) = 2(1+3x)$$

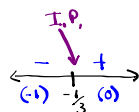
$$0 = 2(1+3x)$$

$$2 \neq 0 \quad | \quad 1+3x = 0$$

$$3x = -1$$

$$x = -\frac{1}{3}$$

$$CV: x = -\frac{1}{3}$$

CD on $(-\infty, -\frac{1}{3})$ CU on $(-\frac{1}{3}, \infty)$ ⑦ Inflection Point: @ $x = -\frac{1}{3}$

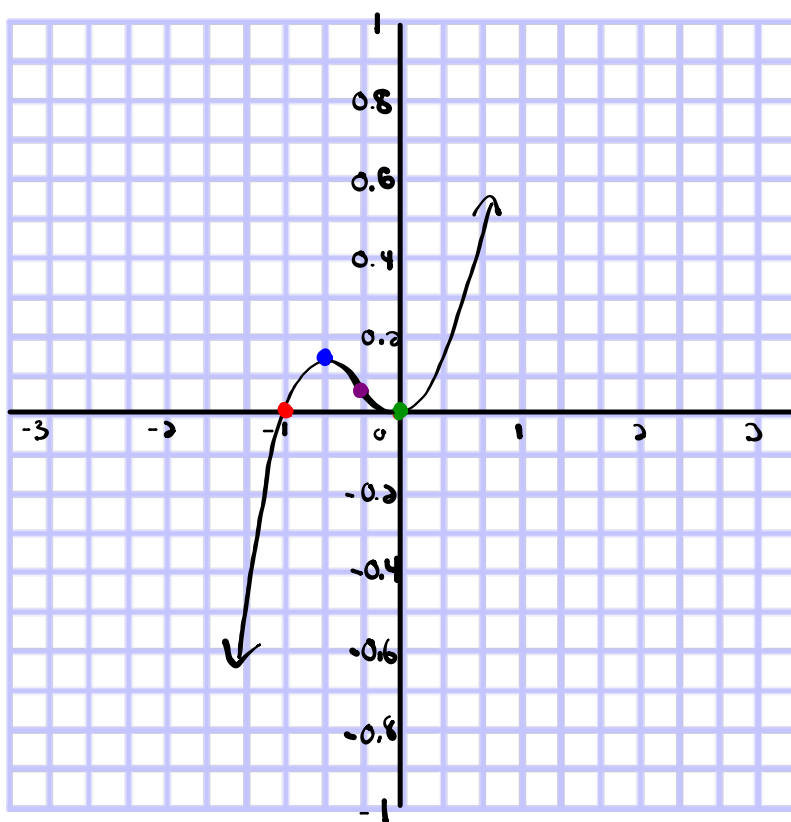
$$f(x) = x^2 + x^3$$

$$f(-\frac{1}{3}) = (-\frac{1}{3})^2 + (-\frac{1}{3})^3$$

$$f(-\frac{1}{3}) = \frac{1}{9} - \frac{1}{27}$$

$$f(-\frac{1}{3}) = \frac{3}{27} - \frac{1}{27} = \frac{2}{27}$$

$$(-\frac{1}{3}, \frac{2}{27}) \text{ or } (-0.3, 0.07)$$



homework

Examine the function $f(x) = \frac{x^2}{1-x^2}$ with respect to... $f'(x) = \frac{2x}{(1-x^2)^2}$

- Intercepts $f(x)$
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

$$f''(x) = \frac{2(1+3x^2)}{(1-x^2)^3}$$

ⓐ x-int ($y=0$)

$$f(x) = \frac{x^2}{1-x^2}$$

$$0 = \frac{x^2}{1-x^2}$$

$$0 = x^2$$

$$0 = x$$

$$(0,0)$$

ⓑ y-int ($x=0$)

$$f(x) = \frac{x^2}{1-x^2}$$

$$f(0) = \frac{0^2}{1-0^2}$$

$$f(0) = \frac{0}{1} = 0$$

$$(0,0)$$

ⓒ Vertical Asymptote: (zeros of the denominator)

$$f(x) = \frac{x^2}{1-x^2}$$

$$\text{VA: } 1-x^2=0$$

$$(1-x)(1+x)=0$$

$$1-x=0 \quad 1+x=0$$

$$1=x \quad x=-1$$

ⓓ Horizontal Asymptote:

$$\lim_{x \rightarrow \infty} \frac{x^2}{1-x^2} = \frac{1}{-1} = -1$$

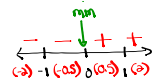
$$y = -1$$

ⓔ Intervals of Inc/Dec.

$$f'(x) = \frac{2x}{(1-x^2)^2}$$

$$\begin{array}{l} 2x=0 \\ x=0 \end{array} \left| \begin{array}{l} (1-x^2)^2=0 \\ 1-x^2=0 \\ 1-x^2 \\ \pm 1=x \end{array} \right.$$

$$\text{Cr: } x = -1, 0, 1$$



Increasing on $(0, \infty)$
Decreasing on $(-\infty, 0)$

ⓕ min @ $x=0$

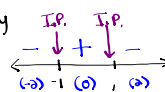
$$f(x) = \frac{x^2}{1-x^2}$$

$$f(0) = \frac{0^2}{1-0^2} = 0$$

$$(0,0)$$

ⓖ Intervals of concavity

$$f''(x) = \frac{2(1+3x^2)}{(1-x^2)^3}$$



$$\begin{array}{l} 2(1+3x^2)=0 \\ 1+3x^2=0 \\ 3x^2=-1 \\ x^2=-\frac{1}{3} \end{array} \left| \begin{array}{l} (1-x^2)^3=0 \\ 1-x^2=0 \\ 1-x^2 \\ \pm 1=x \end{array} \right.$$

Not Possible

(Numerical is always positive)

CO on $(-\infty, -1) \cup (1, \infty)$
CU on $(-1, 1)$

ⓗ Inflection Points:

when $x=-1$

$$f(x) = \frac{x^2}{1-x^2}$$

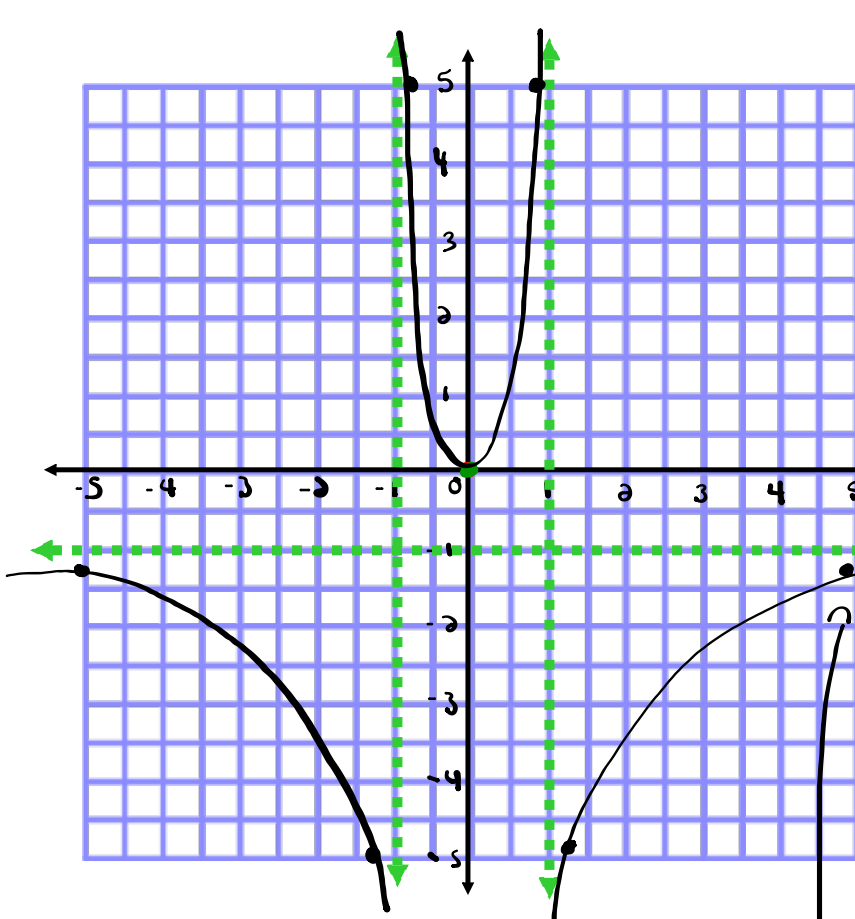
$$f(-1) = \frac{(-1)^2}{1-(-1)^2} = \frac{1}{0} = \text{und.}$$

when $x=1$

$$f(x) = \frac{x^2}{1-x^2}$$

$$f(1) = \frac{(1)^2}{1-(1)^2} = \frac{1}{0} = \text{und.}$$

* No Inflection Points $x = \pm 1$ are the Vertical Asymptotes.



$$f(x) = \frac{x^2}{(1-x)(1+x)}$$

$$\lim_{x \rightarrow 1^-} f(x) = \frac{(+)}{(+)(-)} = -\infty$$

(x = -1.1)

$$\lim_{x \rightarrow 1^+} f(x) = \frac{(+)}{(+)(+)} = +\infty$$

(x = -0.9)

$$\lim_{x \rightarrow 1^-} f(x) = \frac{(+)}{(+)(+)} = +\infty$$

(x < 0.9)

$$\lim_{x \rightarrow 1^+} f(x) = \frac{(+)}{(-)(+)} = -\infty$$

(x = 1.1)

$$f(5) = \frac{(5)^2}{1-(5)^2} = \frac{25}{-24}$$

$$f(x) = \frac{x^2}{1-x^2}$$

$$f'(x) = \frac{2x(1-x^2) + 2x(x^2)}{(1-x^2)^2}$$

$$f'(x) = \frac{2x - 2x^3 + 2x^3}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

$$f''(x) = \frac{2(1-x^2)^2 - 2x(2)(1-x^2)(-2x)}{(1-x^2)^4}$$

$$f''(x) = \frac{2(1-x^2)^2 + 8x^2(1-x^2)}{(1-x^2)^4}$$

$$f''(x) = \frac{2(1-x^2)[(1-x^2) + 4x^2]}{(1-x^2)^4} =$$

$$f''(x) = \frac{2(1-x^2)(1+3x^2)}{(1-x^2)^4} = \frac{2(1+3x^2)}{(1-x^2)^3}$$

Review:

① $f(x) = \frac{8(x-2)}{x^2}$ $f'(x) = \frac{-8(x-4)}{x^3}$ $f''(x) = \frac{16(x-6)}{x^4}$

① Intercepts:

x-int (y=0)

$\frac{0}{1} = \frac{8(x-2)}{x^2}$

$8(x-2) = 0$

$x-2 = 0$

$x = 2$

$(2, 0)$

y-int (x=0)

$f(0) = \frac{8(0-2)}{(0)^2}$

$f(0) = \frac{-16}{0}$

$f(0) = \text{undefined}$

No y-int

② Symmetry:

$f(x) = \frac{8(x-2)}{x^2} = \frac{8x-16}{x^2}$

$f(-x) = \frac{8(-x-2)}{(-x)^2}$

$f(-x) = \frac{8(-x-2)}{x^2}$

$f(-x) = \frac{-8(x+2)}{x^2} = \frac{-8x-16}{x^2}$

No symmetry

③ VA: (zeros of denom)

$f(x) = \frac{8(x-2)}{x^2}$

$x^2 = 0$

$x = 0$

Check behavior of $f(x)$ near VA:

$\lim_{x \rightarrow 0^-} f(x) = \frac{(-)}{(+)} = -\infty$

(x=0.01)

$\lim_{x \rightarrow 0^+} f(x) = \frac{(-)}{(+)} = -\infty$

(x=0.01)

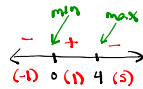
④ HA:

$f(x) = \frac{8(x-2)}{x^2} = \frac{8x-16}{x^2}$

$\lim_{x \rightarrow \infty} \frac{8x-16}{x^2} = 0$

$y = 0$

⑤ Intervals of Inc/Dec



$f'(x) = \frac{-8(x-4)}{x^3}$

Increasing on (0,4)

Decreasing on $(-\infty, 0) \cup (4, \infty)$

CV:

$-8(x-4) = 0$ $x^3 = 0$

$x-4 = 0$ $x = 0$

$x = 4$

⑥ Max/Min:

$f(x) = \frac{8(x-2)}{x^2}$

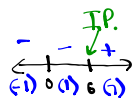
$f(0) = \frac{8(0-2)}{(0)^2} = \text{undefined}$

no min
x=0 is the VA

$f(4) = \frac{8(4-2)}{(4)^2} = \frac{16}{16} = 1$

max at (4,1)

⑦ Concavity:



$f''(x) = \frac{16(x-6)}{x^4}$

CU on (0,6)

CD on $(-\infty, 0) \cup (6, \infty)$

CV:

$16(x-6) = 0$ $x^4 = 0$

$x-6 = 0$ $x = 0$

$x = 6$

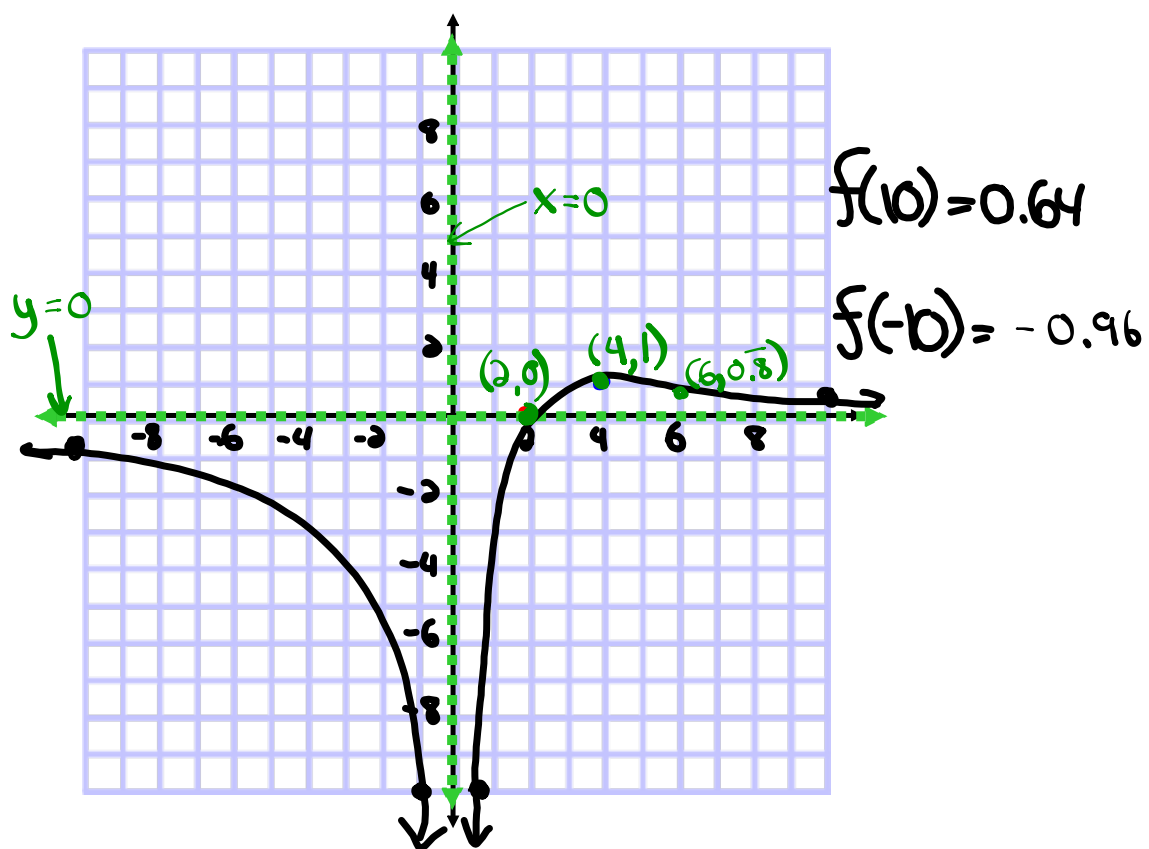
⑧ Inflection Point

$f(x) = \frac{8(x-2)}{x^2}$

$f(6) = \frac{8(6-2)}{6^2} = \frac{32}{36}$

$(6, 0.\bar{8})$

Review:



Review:

Examine the function $f(x) = x^4 - 4x^3$ with respect to...

- Intercepts
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

$$f(x) = x^4 - 4x^3 \quad f'(x) = 4x^3 - 12x^2 \quad f''(x) = 12x^2 - 24x$$

$$f(x) = x^3(x-4) \quad f'(x) = 4x^2(x-3) \quad f''(x) = 12x(x-2)$$

① x-int ($y=0$)

$$f(x) = x^3(x-4)$$

$$0 = x^3(x-4)$$

$$x^3 = 0 \quad | \quad x-4 = 0$$

$$x = 0 \quad | \quad x = 4$$

$$(0,0) \quad | \quad (4,0)$$

② y-int ($x=0$)

$$f(x) = x^4 - 4x^3$$

$$f(0) = (0)^4 - 4(0)^3$$

$$f(0) = 0$$

$$(0,0)$$

③ Intervals of Inc/Dec:

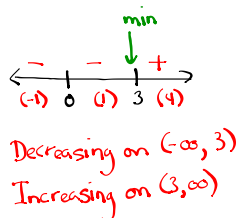
$$f'(x) = 4x^2(x-3)$$

$$4x^2 = 0 \quad | \quad x-3 = 0$$

$$x^2 = 0 \quad | \quad x = 3$$

$$x = 0 \quad | \quad x = 3$$

$$CV: x = 0, 3$$



④ min @ $x=3$

$$f(x) = x^4 - 4x^3$$

$$f(3) = (3)^4 - 4(3)^3$$

$$f(3) = 81 - 108$$

$$f(3) = -27$$

$$(3, -27)$$

⑤ No local max

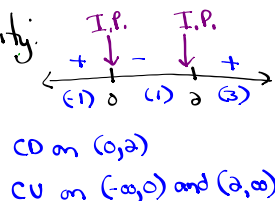
⑥ Intervals of concavity:

$$f''(x) = 12x(x-2)$$

$$12x = 0 \quad | \quad x-2 = 0$$

$$x = 0 \quad | \quad x = 2$$

$$CV: x = 0, 2$$



⑦ Inflection points:

when $x=0$

$$f(x) = x^4 - 4x^3$$

$$f(0) = (0)^4 - 4(0)^3$$

$$f(0) = 0$$

$$(0,0)$$

when $x=2$

$$f(x) = x^4 - 4x^3$$

$$f(2) = (2)^4 - 4(2)^3$$

$$f(2) = 16 - 32$$

$$f(2) = -16$$

$$(2, -16)$$

Review:

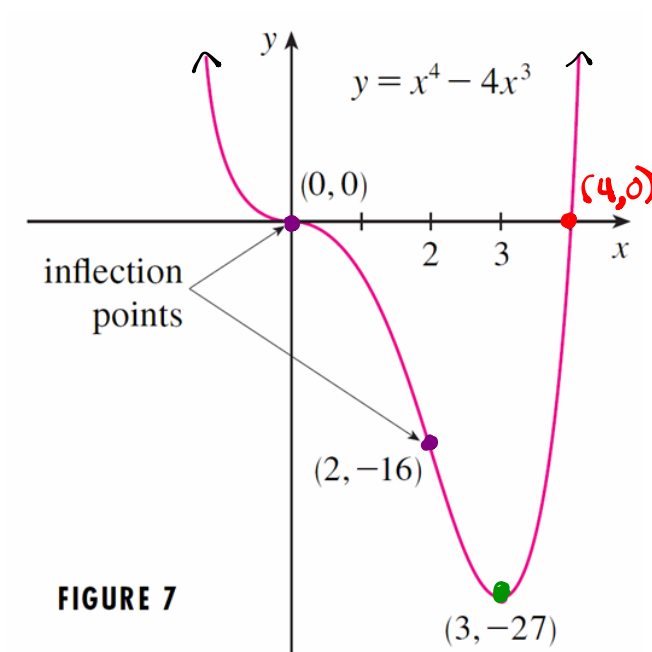


FIGURE 7

homework

Examine the function $f(x) = \frac{x^2}{x-7}$ with respect to ...

- Intercepts
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

① x-int (y=0) ② y-int (x=0) ③ Symmetry:

$$f(x) = \frac{x^2}{x-7}$$

(x-1) $0 = \frac{x^2}{x-7}$ (x-7) $f(x) = \frac{x^2}{x-7}$ $f(x) = \frac{x^2}{x-7}$

$$0 = \frac{x^2}{x-7}$$

$$0 = x^2$$

$$0 = x$$

(0,0)

$f(0) = \frac{0^2}{0-7} = 0$

(0,0)

$f(-x) = \frac{(-x)^2}{-x-7} = \frac{x^2}{-x-7}$

No symmetry

④ VA: (denom=0)

$$x-7=0$$

$$x=7$$

$\lim_{x \rightarrow 7^-} \frac{x^2}{x-7} = \frac{49}{0^-} = -\infty$ $\lim_{x \rightarrow 7^+} \frac{x^2}{x-7} = \frac{49}{0^+} = +\infty$

(6,9) (6,9)

⑤ SA: $\frac{x-7}{x^2}$ $y = x+7$ $m = \frac{1}{1}$ rise
run

$-\frac{(x^2-7x)}{49}$ $b = 7$ y.int

$-\frac{(7x-49)}{49}$

⑥ Intervals of Inc/Dec:

$f'(x) = \frac{x(x-14)}{(x-7)^2}$

max neither min

+ - +

(-1) 0 (7) (14) (5)

CV: $x=0$ | $x-14=0$ | $(x-7)^2=0$

$x=14$ | $x-7=0$ | $x=7$

Increasing on $(-\infty, 0) \cup (14, \infty)$

$x < 0 + x > 14$

Decreasing on $(0, 14)$

$0 < x < 14$

⑦ Local max/min

$f(x) = \frac{x^2}{x-7}$

When $x=0$ When $x=14$

$f(0) = \frac{0^2}{0-7} = 0$ $f(14) = \frac{14^2}{14-7} = \frac{196}{7} = 28$

(0,0) (14,28)

local max @ (0,0) local min @ (14,28)

⑧ Intervals of Concavity:

$f''(x) = \frac{98}{(x-7)^3}$

- I.P. +

(6) (7) (8)

CV: $98 \neq 0$ | $(x-7)^3=0$ Concave down on $(-\infty, 7)$

$x-7=0$ $x < 7$

$x=7$ Concave up on $(7, \infty)$

$x > 7$

⑨ I.P. (x=7)

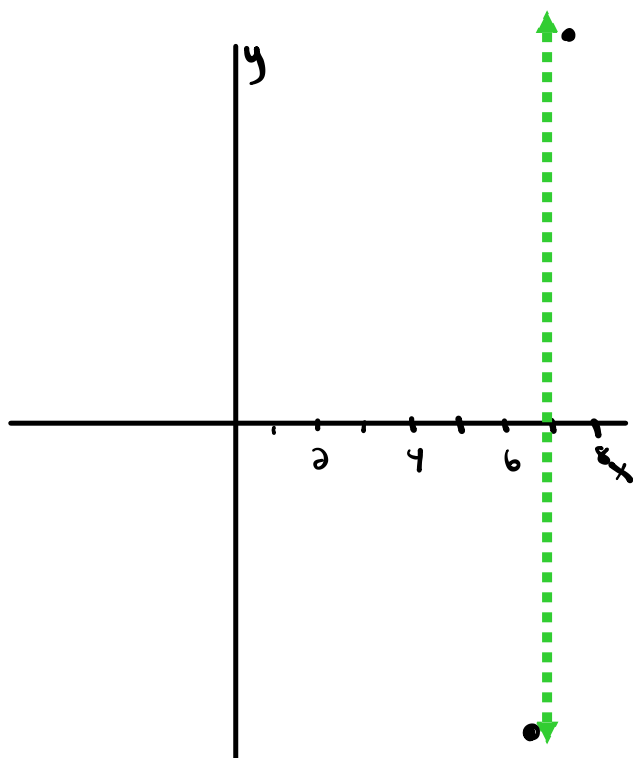
$f(x) = \frac{x^2}{x-7}$ $x=7$ is the vertical asymptote

$f(7) = \frac{7^2}{7-7} = \frac{49}{0} = \text{DNE}$

$$f(x) = \frac{x^2}{x-7}$$

$$\text{VA: } x-7=0$$

$$x=7$$



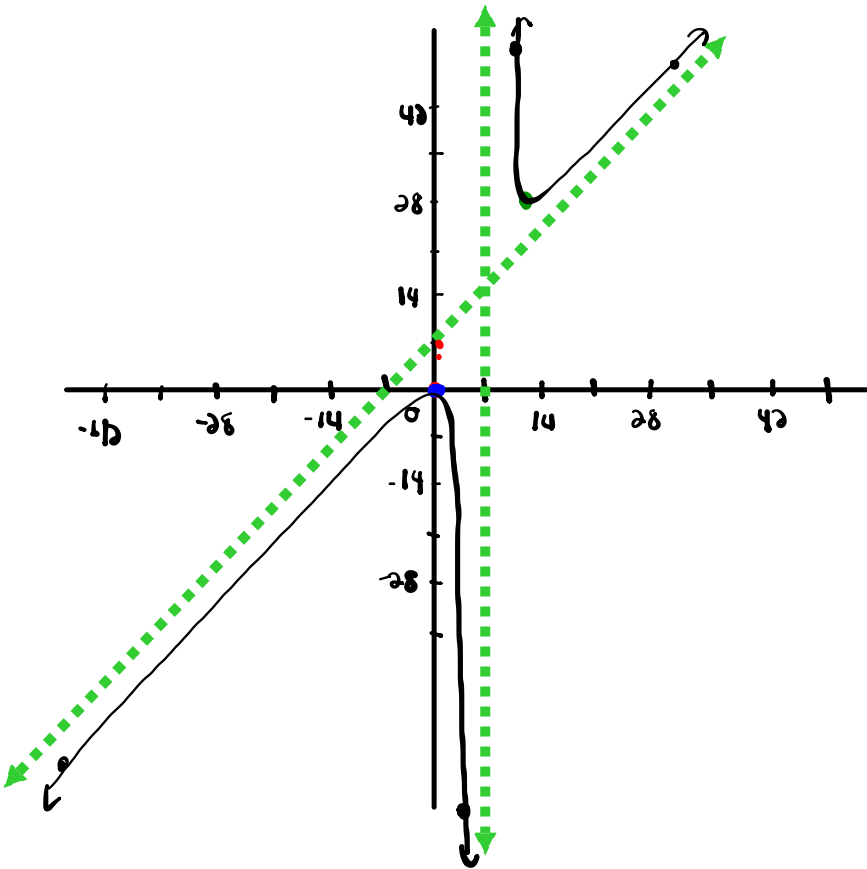
Check Behaviour near VA

$$\lim_{x \rightarrow 7^-} \frac{(+)}{(-)} = -\infty$$

(x=6.9)

$$\lim_{x \rightarrow 7^+} \frac{(+)}{(+)} = +\infty$$

(x=7.1)



Review:

$$f(x) = x^4 - 3x^3 + 3x^2 - x$$

$$(1)^3 - 3(1)^2 + 3(1) - 1$$

$$1 - 3 + 3 - 1$$

① x-int (y=0)

$$0 = x^4 - 3x^3 + 3x^2 - x$$

$$0 = x(x^3 - 3x^2 + 3x - 1) \text{ syn. sub.}$$

$$0 = x(x-1)(x^2 - 2x + 1) \text{ simple tri.}$$

$$\frac{-1}{-1} x \frac{-1}{-1} = 1$$

$$\frac{-1}{-1} + \frac{-1}{-1} = -2$$

$$0 = x(x-1)(x-1)(x-1)$$

$$x=0 \quad | \quad x-1=0$$

$$(0,0) \quad | \quad x=1$$

$$(1,0)$$

$$\Downarrow \quad \begin{array}{cccc} 1 & -3 & 3 & -1 \\ & 1 & -2 & 1 \\ \hline 1 & -2 & 1 & \end{array}$$

② y-int (x=0)

$$f(x) = x^4 - 3x^3 + 3x^2 - x$$

$$f(0) = (0)^4 - 3(0)^3 + 3(0)^2 - (0) = 0$$

$$(0,0)$$

Intervals of Inc/Dec:

$$f(x) = x^4 - 3x^3 + 3x^2 - x$$

$$f'(x) = 4x^3 - 9x^2 + 6x - 1 \text{ syn. sub.}$$

$$f'(x) = (x-1)(4x^2 - 5x + 1) \text{ Hard - Trinomial}$$

$$f'(x) = (x-1)(x-\frac{1}{4})(x-\frac{1}{4})$$

$$\frac{-1}{-1} x \frac{-4}{-4} = 4$$

$$\frac{-1}{-1} + \frac{-4}{-4} = -5$$

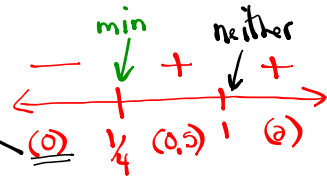
$$f'(x) = (x-1)(4x-1)(x-1)$$

$$f'(x) = (x-1)^2(4x-1)$$

$$4(1)^3 - 9(1)^2 + 6(1) - 1$$

$$4 - 9 + 6 - 1$$

$$\Downarrow \quad \begin{array}{cccc} 4 & -9 & 6 & -1 \\ & 4 & -5 & 1 \\ \hline 4 & -5 & 1 & \end{array}$$



$$cv: \quad (x-1)^2 = 0 \quad | \quad 4x-1 = 0$$

$$x-1 = 0 \quad | \quad 4x = 1$$

$$x = 1 \quad | \quad x = \frac{1}{4}$$

Decreasing on $(-\infty, \frac{1}{4})$
 $x < \frac{1}{4}$

$$x = \frac{1}{4}, 1$$

Increasing on $(\frac{1}{4}, \infty)$
 $x > \frac{1}{4}$