Curve Sketching

In this chapter we look at further aspects of curves such as vertical and horizontal asymptotes, concavity, and inflections points. Then we use them, together with intervals of increase and decrease and maximum and minimum values, to develop a procedure for curve sketching.

Questions from Homework

① a) $x=-7$, $x=-3$, $x=2$, $x=6$

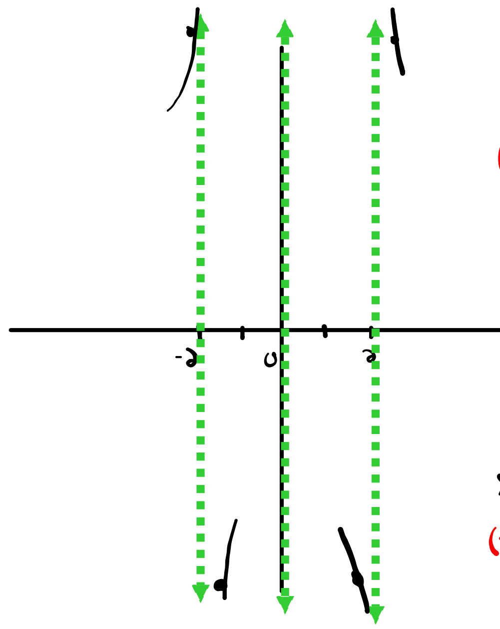
b) (i) $\lim\limits_{x \to -7^-} f(x) = -\infty$

(ii) $\lim\limits_{x \to -7^+} f(x) = \infty$

③ h) $$y = \frac{1}{x^4-4x^2} = \frac{1}{x^2(x^2-4)} = \frac{1}{x^2(x-2)(x+2)}$$

VA: $x^2(x+2)(x-2)=0$

$x^2=0$	$x+2=0$	$x-2=0$
$\boxed{x=0}$	$\boxed{x=-2}$	$\boxed{x=2}$



$$\lim_{x \to 2^-} \frac{1}{x^2(x+2)(x-2)} = \frac{(+)}{(+)(+)(-)} = -\infty$$
(x=1.9)

$$\lim_{x \to 2^+} \frac{(+)}{(+)(+)(+)} = +\infty$$
(x=2.1)

$$\lim_{x \to -2^-} \frac{(+)}{(+)(-)(-)} = +\infty$$
(x=-2.1)

$$\lim_{x \to -2^+} \frac{(+)}{(+)(+)(-)} = -\infty$$
(x=-1.9)

Questions from Homework

③ f) $y = \dfrac{6x^3}{x^2+4x+3} = \dfrac{6x^3}{(x+1)(x+3)}$

VA:

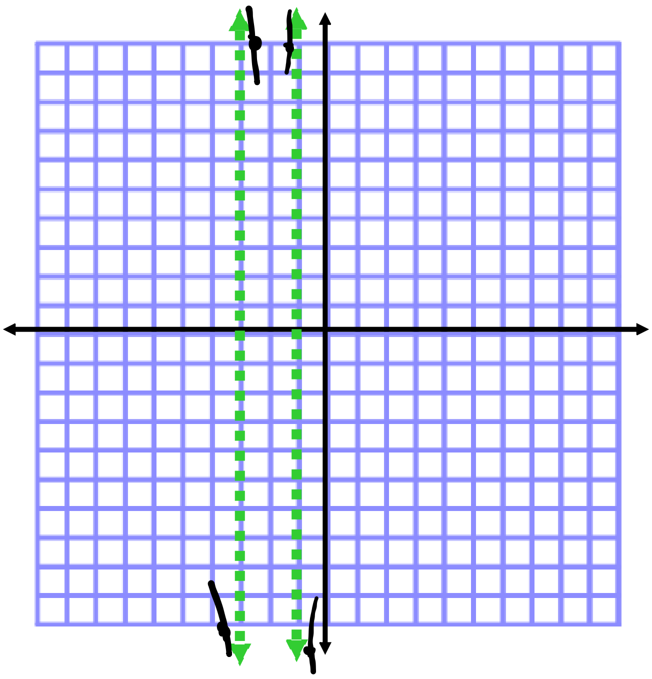
$x^2+4x+3=0$

$(x+1)(x+3)=0$

$x+1=0$ | $x+3=0$

$\boxed{x=-1}$ | $\boxed{x=-3}$

$y = \dfrac{6x^3}{(x+1)(x+3)}$



$\lim_{x\to -3^-} f(x) = \dfrac{(-)}{(-)(-)} = \dfrac{(-)}{(+)} = -\infty$

$(x=-3.1)$

$\lim_{x\to -3^+} f(x) = \dfrac{(-)}{(-)(+)} = \dfrac{(-)}{(-)} = +\infty$

$(x=-2.9)$

$\lim_{x\to -1^-} f(x) = \dfrac{(-)}{(-)(+)} = \dfrac{(-)}{(-)} = +\infty$

$(x=-1.1)$

$\lim_{x\to -1^+} f(x) = \dfrac{(-)}{(+)(+)} = \dfrac{(-)}{(+)} = -\infty$

$(x=-0.9)$

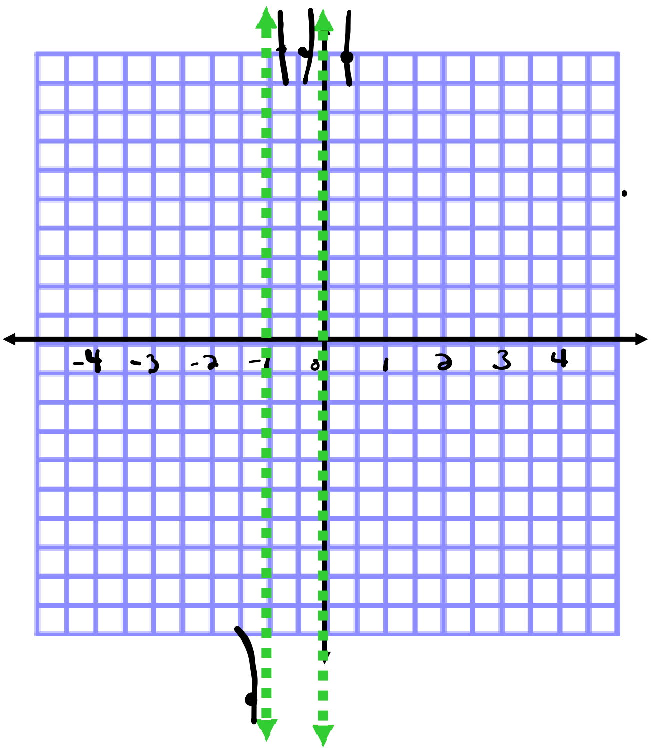
Questions from Homework

③ g) $y = \dfrac{1}{x^2(x+1)}$

VA: $x^2(x+1)=0$

$x^2 = 0$ | $x+1=0$

$\boxed{x=0}$ | $\boxed{x=-1}$



$\lim\limits_{x\to -1^-} \dfrac{(+)}{(+)(-)} = \dfrac{(+)}{(-)} = -\infty$

$(x=-1.1)$

$\lim\limits_{x\to -1^+} \dfrac{(+)}{(+)(+)} = \dfrac{(+)}{(+)} = +\infty$

$(x=-0.9)$

$\lim\limits_{x\to 0^-} \dfrac{(+)}{(+)(+)} = \dfrac{(+)}{(+)} = +\infty$

$(x=-0.1)$

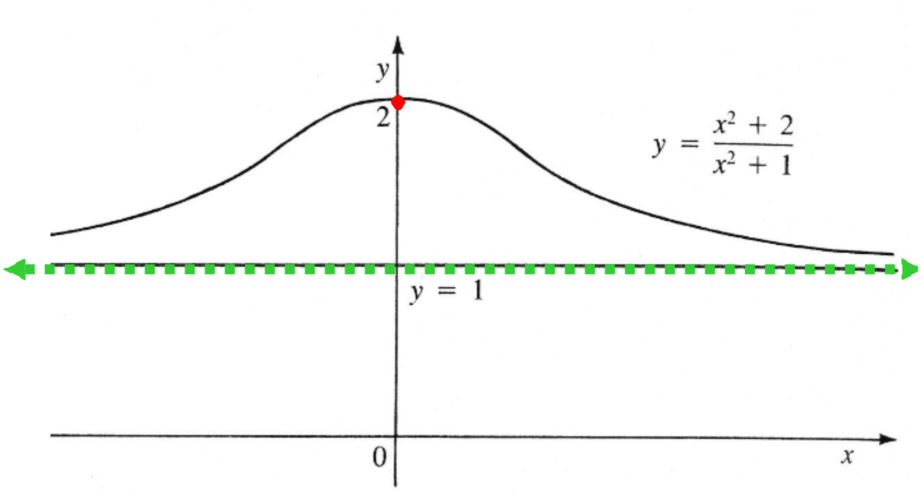
$\lim\limits_{x\to 0^+} \dfrac{(+)}{(+)(+)} = \dfrac{(+)}{(+)} = +\infty$

$(x=0.1)$

Horizontal Asymptotes

Let us examine the behaviour of the function $y = \frac{x^2+2}{x^2+1}$ for large *x values*.

	$f(x)$
0	2
± 1	1.5
± 2	1.2
± 3	1.1
± 4	1.0588
± 5	1.0385
± 50	1.0004
± 100	1.0001



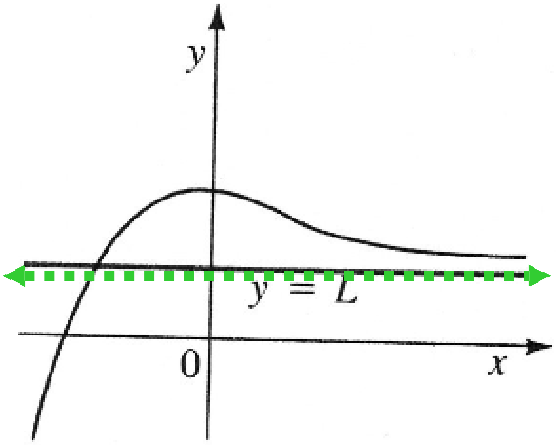
From the table and the graph we see that the *y* values become closer and closer to 1 as *x* gets larger and larger. In fact we can make *y* as close to 1 as we like by taking *x* large enough.

and we say that the line *y* = *1* is a horizontal asymptote of $f(x)$

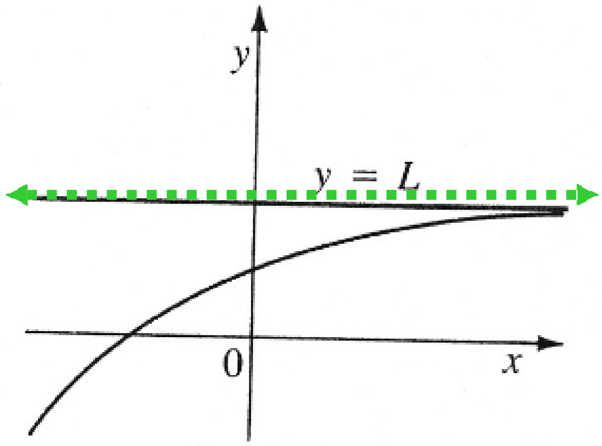
Horizontal Asymptote

The line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either

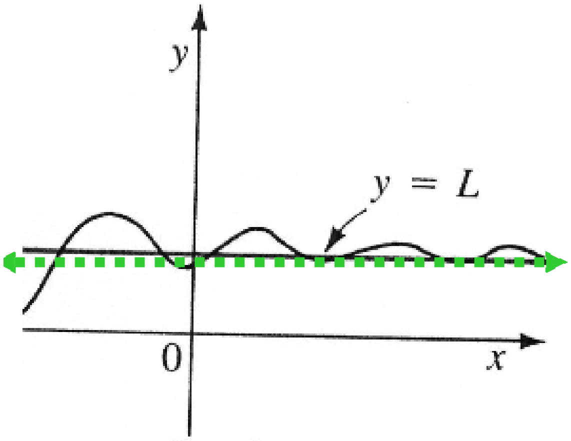
$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$



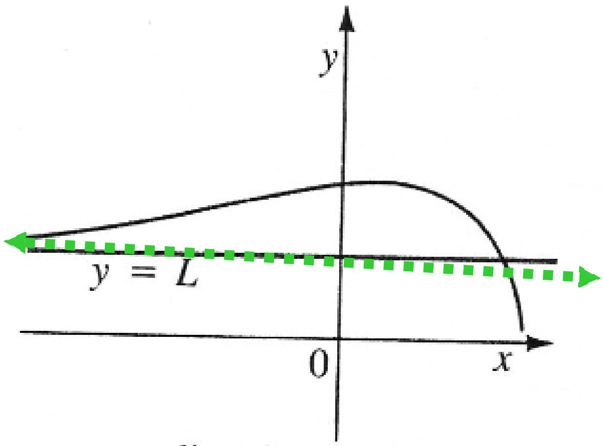
$\lim_{x\to\infty} f(x) = L$



$\lim_{x\to\infty} f(x) = L$



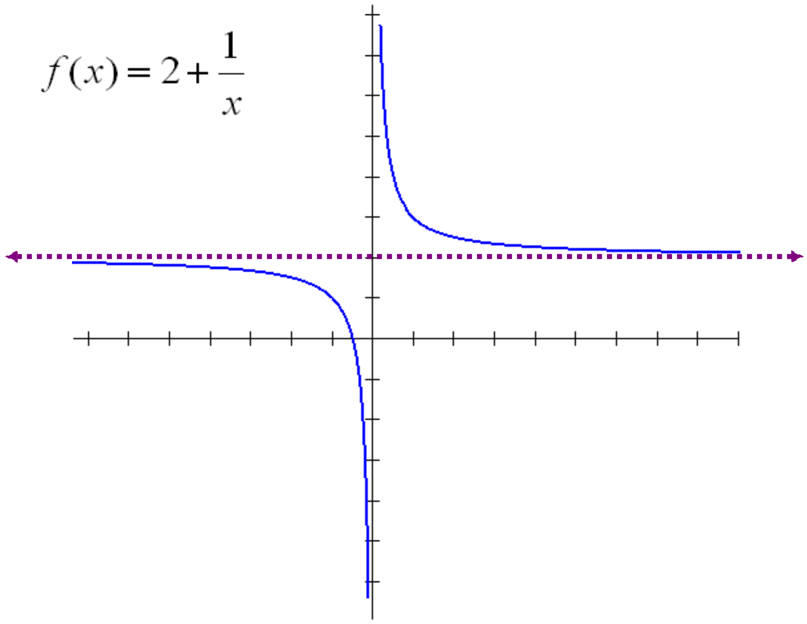
$\lim_{x\to\infty} f(x) = L$



$\lim_{x\to-\infty} f(x) = L$

Examine the limits of $f(x)$ as x approaches $\pm\infty$

$$f(x) = 2 + \frac{1}{x}$$

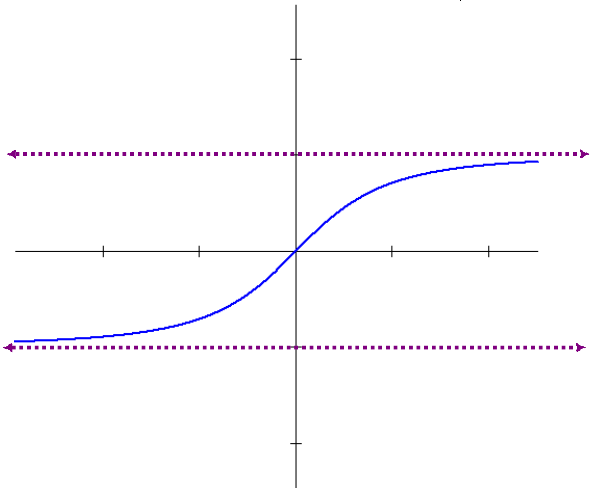


$$\lim_{x\to\infty} f(x) = 2$$

$$\lim_{x\to-\infty} f(x) = 2$$

There can be more than one horizontal asymptote.

Examine the function $f(x) = \dfrac{x}{\sqrt{x^2+1}}$



$\lim\limits_{x\to\infty} f(x) = 1$

$\lim\limits_{x\to-\infty} f(x) = -1$

Examine the limits of $f(x)$ as x approaches $\pm\infty$

by definition $\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}}$ $\qquad$ $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2+1}}$

$\lim\limits_{x\to\infty} \dfrac{\frac{x}{\sqrt{x^2}}}{\frac{\sqrt{x^2+1}}{\sqrt{x^2}}}$ $\qquad$ $\lim\limits_{x\to-\infty} \dfrac{\frac{x}{\sqrt{x^2}}}{\frac{\sqrt{x^2+1}}{\sqrt{x^2}}}$

$\lim\limits_{x\to\infty} \dfrac{\frac{x}{|x|}}{\sqrt{\cancel{\frac{x^2}{x^2}}+\frac{1}{x^2}}}$ $\qquad$ $\lim\limits_{x\to-\infty} \dfrac{\frac{x}{|x|}}{\sqrt{\cancel{\frac{x^2}{x^2}}+\frac{1}{x^2}}}$

$\lim\limits_{x\to\infty} \dfrac{\frac{\cancel{x}}{\cancel{x}}}{\sqrt{1+\frac{1}{x^2}}}$ $\qquad$ $\lim\limits_{x\to-\infty} \dfrac{\frac{\cancel{x}}{-\cancel{x}}}{\sqrt{1+\frac{1}{x^2}}}$

$\lim\limits_{x\to\infty} \dfrac{1}{\sqrt{1+\frac{1}{x^2}}}$ approaches 0 $\qquad$ $\lim\limits_{x\to-\infty} \dfrac{-1}{\sqrt{1+\frac{1}{x^2}}}$ approaches 0

$= \dfrac{1}{\sqrt{1+0}}$ $\qquad$ $= \dfrac{-1}{\sqrt{1+0}}$

$= 1$ $\qquad$ $= -1$

- Remember

If the highest degree is in the denominator then the *Horizontal asymptote* will be $y = 0$

$$y = \frac{x^2+7x+12}{x^3-1} \qquad HA: \lim_{x\to\infty} \frac{x^2+7x+12}{x^3-1} = 0 \qquad \boxed{y=0}$$

If the highest degree is in the numerator then there will be no *Horizontal asymptote*.

$$y = \frac{x^2+x-6}{x-2} \qquad HA: \lim_{x\to\infty} \frac{x^2+x-6}{x-2} = DNE \qquad \boxed{\text{No HA}}$$

If the degree is the same in the numerator and denominator then the *Horizontal asymptote* will be equal to the quotients of the coefficients in front of the highest degree.

$$f(x) = \frac{1-5x^2}{3x^2+6x+5} \qquad HA: \lim_{x\to\infty} \frac{1-5x^2}{3x^2+6x+5} = \frac{-5}{3} \qquad \boxed{y = -\frac{5}{3}}$$

$$\lim_{x\to\infty} \frac{4x^3-1}{3x^3+2x^2-3} = \frac{4}{3}$$ $\boxed{HA: y = \frac{4}{3}}$

$$= \lim_{x\to\infty} \frac{\frac{4\cancel{x^3}}{\cancel{x^3}} - \frac{1}{x^3}}{\frac{3\cancel{x^3}}{\cancel{x^3}} + \frac{2x^2}{x^3} - \frac{3}{x^3}}$$

$$= \lim_{x\to\infty} \frac{4 - \frac{1}{x^3}}{3 + \frac{2}{x} - \frac{3}{x^3}} = \frac{4-0}{3+0-0} = \frac{4}{3}$$

$$\lim_{x\to\infty} \frac{x^2+7x+12}{x^3-1} = 0$$ $\boxed{HA: y = 0}$

$$= \lim_{x\to\infty} \frac{\frac{x^2}{x^3} + \frac{7x}{x^3} + \frac{12}{x^3}}{\frac{x^3}{x^3} - \frac{1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x} + \frac{7}{x^2} + \frac{12}{x^3}}{1 - \frac{1}{x^3}} = \frac{0+0+0}{1-0} = 0$$

$$\lim_{x\to\infty} \frac{x^3-1}{x+3} = \text{DNE}$$ $\boxed{\text{HA: none}}$

$$\lim_{x\to\infty} \frac{\frac{x^3}{x^3} - \frac{1}{x^3}}{\frac{x}{x^3} + \frac{3}{x^3}}$$

$$\lim_{x\to\infty} \frac{1 - \frac{1}{x^3}}{\frac{1}{x^2} + \frac{3}{x^3}} = \frac{1-0}{0+0} = \text{DNE}$$

Example

Find the horizontal and vertical asymptotes of $y = \dfrac{x}{x^2 - x - 6}$

and sketch the graph

HA: $\lim\limits_{x\to\infty} \dfrac{x}{x^2-x-6} = 0$

$\boxed{y=0}$

VA: $x^2-x-6=0$

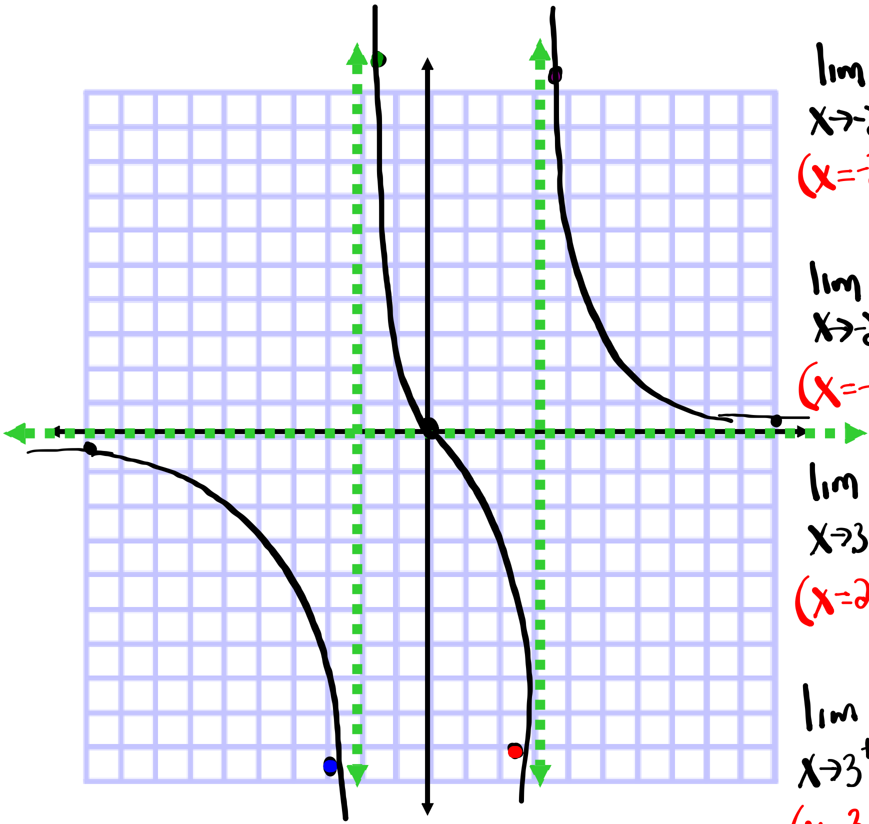
$-3 \times 2 = -6$

$-3 + 2 = -1$

$(x-3)(x+2)=0$

$x-3=0$ | $x+2=0$

$\boxed{x=3}$ | $\boxed{x=-2}$



$\lim\limits_{x\to -2^-} \dfrac{x}{(x-3)(x+2)} = \dfrac{(-)}{(-)(-)} = -\infty$

$(x=-2.1)$

$\lim\limits_{x\to -2^+} \dfrac{x}{(x-3)(x+2)} = \dfrac{(-)}{(-)(+)} = +\infty$

$(x=-1.9)$

$\lim\limits_{x\to 3^-} \dfrac{x}{(x-3)(x+2)} = \dfrac{(+)}{(-)(+)} = -\infty$

$(x=2.9)$

$\lim\limits_{x\to 3^+} \dfrac{x}{(x-3)(x+2)} = \dfrac{(+)}{(+)(+)} = +\infty$

$(x=3.1)$

Homework

Sketch the following function:

$$f(x)=\frac{8(x-2)}{x^2} \qquad f'(x)=\frac{-8(x-4)}{x^3} \qquad f''(x)=\frac{16(x-6)}{x^4}$$

Be sure to examine...

- Intercepts
- Asymptotes (*vertical and horizontal)*
- Regions of increase/decrease
- Local extrema
- Regions where concave up/down
- Inflection points