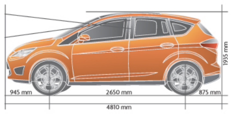
5.5 Optimization Problems II: Exploring Solutions

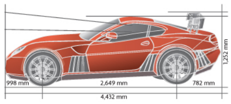
GOAL

Explore the feasible region of a system of linear inequalities.

EXPLORE the Math

A toy company manufactures two types of toy vehicles: sport-utility vehicles and racing cars.





- Because the supply of materials is limited, no more than 40 racing cars and 60 sport-utility vehicles can be made each day.
- However, the company can make 70 or more vehicles, in total, each day.
- It costs \$8 to make a racing car and \$12 to make a sport-utility vehicle.

There are many possible combinations of racing cars and sport-utility vehicles that could be made. The company wants to know what combinations will result in the minimum and maximum costs, and what those costs will be.

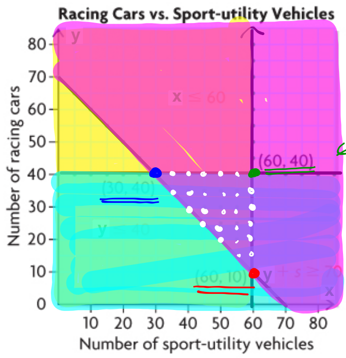
The following model represents this situation. The feasible region of the graph represents all the possible combinations of racing cars y and sport-utility vehicles x.

Variables:
Let x represent the number of sport-utility vehicles.
Let y represent the number of racing cars.
Let C represent the cost of production.

Restrictions:
$x \in W$, $y \in W$

Constraints:

$y \leq 40$
$x \leq 60$
$y + x \geq 70$



max cost

Objective function to optimize: → maximize / minimize
$C = 12x + 8y$

How can you use patterns in the feasible region to predict the combinations of sport-utility vehicles and racing cars that will result in the minimum and maximum values of the objective function? → solution set

Value of $C = 8r + 12s$ or $C = 8y + 12x$

- as x increases: 720 at (40, 30), 840 at (50, 30), 960 at (60, 30)
- as y increases: 860 at (45,40), 780 at (45, 30) 740 at (45, 25)
- in the middle of the solution region: 820 at (45, 35), 880 at (50, 35), 840 at (50, 30)
- at the corners of the solution region: 800 at (60, 10), 1040 at (60, 40), 680 at (30, 40) where the lines intersect

objective function:

$C = 12x + 8y$
$C = 12(40) + 8(30)$
$C = 480 + 240$
$\boxed{C = \$720}$

(30,40)
$C = 12x + 8y$
$C = 12(30) + 8(40)$
$C = 360 + 320$
$C = \$680$
minimum cost.

(60,10)
$C = 12(60) + 8(10)$
$C = 720 + 80$
$C = \$800$

(60,40)
$C = 12(60) + 8(40)$
$C = 720 + 320$
$C = \$1040$ maximum cost

How can you use patterns in the feasible region to predict the combinations of sport-utility vehicles and racing cars that will result in the minimum and maximum values of the objective function?

Sample Solution

Value of $C = 8y + 12x$

- as x increases: 720 at (40, 30), 840 at (50, 30), 960 at (60, 30)
- as y increases: 860 at (45,40), 780 at (45, 30) 740 at (45, 25)
- in the middle of the solution region: 820 at (45, 35), 880 at (50, 35), 840 at (50, 30)
- at the corners of the solution region: 800 at (60, 10), 1040 at (60, 40), 680 at (30, 40)

Reflecting

A. With a partner, discuss the pattern in the value of C throughout the feasible region. Is the pattern what you expected? Explain.

B. As you move from left to right across the feasible region, what happens to the value of C?

C. As you move from the bottom to the top of the feasible region, what happens to the value of C?

D. What points in the feasible region result in each **optimal solution**?
i) the maximum possible value of C
ii) the minimum possible value of C

E. Explain how you could verify that your solutions from part D satisfy each constraint in the model.

optimal solution
A point in the solution set that represents the maximum or minimum value of the objective function.

Answers

A. I noticed that the maximum and minimum values were found at two of the corners. This makes sense, since this is where the values of the coordinates are the greatest or least.

max and mins @ vertices

B. As the value of x increases, so does the value of C.

C. As the value of y increases, so does the value of C.

D. i) The maximum value of C is at (60, 40), where C is 1040.

ii) The minimum value of C is at (30, 40), where C is 680.

E. i) (60, 40)

(x, y) is (60, 40)
$y \leq 40$

LS	RS
40	40
$40 \leq 40$	

(x, y) is (60, 40)
$x \leq 60$

LS	RS
60	60
$60 \leq 60$	

(x, y) is (60, 40)
$y + x \geq 70$

LS	RS
40 + 60 100	70
$100 \geq 70$	

ii) (30, 40)

(x, y) is (30, 40)
$y \leq 40$

LS	RS
40	40
$40 \leq 40$	

(x, y) is (30, 40)
$x \leq 60$

LS	RS
30	60
$30 \leq 60$	

(x, y) is (30, 40)
$y + x \geq 70$

LS	RS
40 + 30 70	70
$70 \geq 70$	

In Summary

Key Ideas

- The value of the objective function for a system of linear inequalities varies throughout the feasible region, but in a predictable way.
- The optimal solutions to the objective function are represented by points at the intersections of the boundaries of the feasible region. If one or more of the intersecting boundaries is not part of the solution set, the optimal solution will be nearby.

Need to Know

- You can verify each optimal solution to make sure it satisfies each constraint by substituting the values of its coordinates into each linear inequality in the system.
- The intersection points of the boundaries are called the vertices, or corners, of the feasible region.

Assignment: **page 252 - 253**

Questions 1, 2, 3

***In Questions 3e and 3f, change "cookbooks" to "books".**

SOLUTIONS ⇒ 5.5 Optimization Problems II: Exploring Solutions.

1. Where might you find the maximum and minimum solutions to each objective function below? Explain how you know. corners or vertices

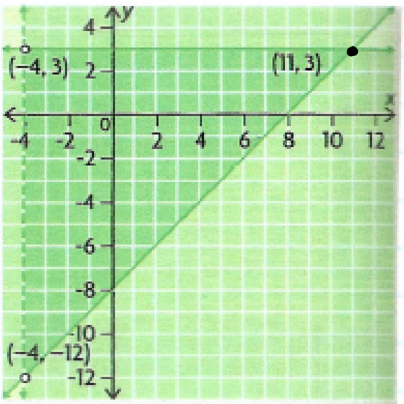
a) Model A

Restrictions:
$x \in R, y \in R$

Constraints:
$x > -4$
$x - y \leq 8$
$y < 3$

Objective function:
$T = 2x + 5y$

Graph of Model A:



⇒ You will find the maximum solution located at (11, 3). (11,3) is closed

⇒ You will find the minimum solution located near (−4, −12).
* (−4, −12) is not included in the feasible region. it is an open dot

b) Model B

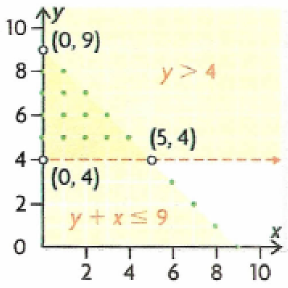
Restrictions:
$x \in W, y \in W$

Constraints:

$y > 4$
$y + x \leq 9$

Objective function:
$N = 3x - 2y$

Graph of Model B:



⇒ You will find the maximum solution located near (5,4).
* (5,4) is not located in the feasible region.

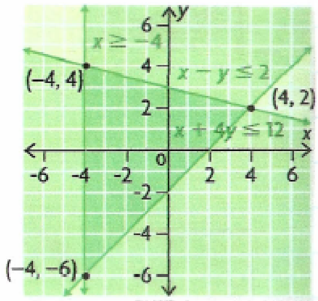
⇒ You will find the minimum solution located near (0,9).
* (0,9) is not located in the feasible region.

2. Consider the model below. What point in the feasible region would result in the minimum value of the objective function? How could you have predicted this from examining the objective function?

Restrictions:
$x \in R, y \in R$

Constraints:
$x + 4y \leq 12$
$x - y \leq 2$
$x \geq -4$

Objective function:
$P = x - y$



⇒ The point in the feasible region that would result in the minimum value of the objective function would be (-4,4).
This is because the objective function is the difference of x and y. The other 2 points would result in a positive difference.

3. Meg is building a bookshelf to display her cookbooks and novels.

- She has no more than 50 cookbooks and no more than 200 novels.
- She wants to display at least 2 novels for every cookbook.
- The cookbook spines are about half an inch wide, and the novel spines are about a quarter of an inch wide.

Meg wants to know how long to make the bookshelf.

The following model represents this situation.

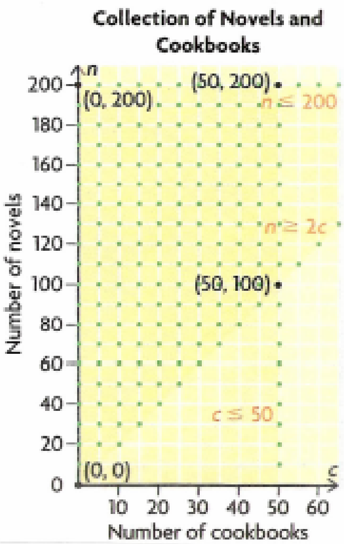
Let c represent the number of cookbooks.
Let n represent the number of novels.
Let W represent the width of the bookshelf.

Restrictions:
$c \in \mathbb{W}$, $n \in \mathbb{W}$

Constraints:

$c \leq 50$
$n \leq 200$
$n \geq 2c$

Objective function:
$W = 0.5c + 0.25n$



0.5 in.

0.25 in.

a) Which point in the feasible region represents the greatest number of books (both cookbooks and novels) that Meg could have? Explain how you know.

=> The point (50,200) represents the greatest number of books that Meg could have. We know this since it is the farthest point from both axes.

b) Can she display the same number of cookbooks and novels? Explain.

=> She cannot display the same number of cookbooks and novels as these points are not located in the feasible region.

c) What point represents the most cookbooks and the fewest novels?

=> (50,100)

d) What point represents the number of BOOKS that would require the longest shelf? How long would the shelf have to be?

=> (50, 200)

$W = 0.5c + 0.25n$
$= 0.5(50) + 0.25(200)$
$= 25 + 50$
$= 75$ inches.

e) What point represents the number of BOOKS that would require the shortest shelf?

=> (0,0) *No shelving would be required at all!