Example 1

Let $F(x) = f(g(x))$ composite function

chain rule

If $f(2) = 3$, $f'(2) = 5$, $g(1) = 2$ and $g'(1) = 4$ find $F'(1)$.

$F'(x) = f'(g(x))g'(x)$

$F'(1) = f'(g(1))g'(1)$

$F'(1) = f'(2)g'(1)$

$F'(1) = (5)(4)$

$F'(1) = 20$

$F(x) = (f \cdot g)(x)$

$F'(x) =$ product rule

$F'(x) = f'(x)g(x) + f(x)g'(x)$

$F(x) = \left(\frac{f}{g}\right)(x)$

$F'(x) =$ quotient rule

$F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Example 2

If $y=u^{10}+u^5+2$, where $u=1-3x^2$, find $\left.\frac{dy}{dx}\right]_{x=1}$

(i) Find $\frac{dy}{du}$:

$y=u^{10}+u^5+2$

$\frac{dy}{du}=10u^9+5u^4$

(ii) Find $\frac{du}{dx}$:

$u=1-3x^2$

$\frac{du}{dx}=-6x$

(iii) Find u when $x=1$

$u=1-3x^2$

$u=1-3(1)^2$

$u=1-3$

$u=-2$

(iv) Find $\left.\frac{dy}{dx}\right]_{x=1}$

$\left.\frac{dy}{dx}\right]_{x=1}=\left[\frac{dy}{du}\right]\cdot\left[\frac{du}{dx}\right]$

$\left.\frac{dy}{dx}\right]_{x=1}=\left[10u^9+5u^4\right]\left[-6x\right]$

$\left.\frac{dy}{dx}\right]_{x=1}=\left[10(-2)^9+5(-2)^4\right]\left[-6(1)\right]$

$\left.\frac{dy}{dx}\right]_{x=1}=\left[-5120+80\right]\left[-6\right]$

$\left.\frac{dy}{dx}\right]_{x=1}=\left[-5040\right]\left[-6\right]$

$\left.\frac{dy}{dx}\right]_{x=1}=30240$

Example 2

If $y = u^{10} + u^5 + 2$, where $u = 1 - 3x^2$, find $\left.\frac{dy}{dx}\right]_{x=1}$

(i) Substitute $u = 1 - 3x^2$:

$y = u^{10} + u^5 + 2$

$y = (1-3x^2)^{10} + (1-3x^2)^5 + 2$

(ii) Find $\frac{dy}{dx}$:

$y = (1-3x^2)^{10} + (1-3x^2)^5 + 2$

$\frac{dy}{dx} = 10(1-3x^2)^9(-6x) + 5(1-3x^2)^4(-6x)$

$\frac{dy}{dx} = -60x(1-3x^2)^9 - 30x(1-3x^2)^4$

(iii) Find $\left.\frac{dy}{dx}\right]_{x=1}$

$\left.\frac{dy}{dx}\right]_{x=1} = -60(1)(1-3(1)^2)^9 - 30(1)(1-3(1)^2)^4$

$\left.\frac{dy}{dx}\right]_{x=1} = -60(-2)^9 - 30(-2)^4$

$\left.\frac{dy}{dx}\right]_{x=1} = 30720 - 480$

$\boxed{\left.\frac{dy}{dx}\right]_{x=1} = 30240}$

Review

① a) $f(x) = \sqrt{x-5}$ $\quad f(x+h) = \sqrt{x+h-5}$

$$f'(x) = \lim_{h \to 0} \left(\frac{\sqrt{x+h-5} - \sqrt{x-5}}{h}\right) \frac{\left(\sqrt{x+h-5} + \sqrt{x-5}\right)}{\left(\sqrt{x+h-5} + \sqrt{x-5}\right)}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{x}+h-\cancel{5} - (\cancel{x}-\cancel{5})}{h\left(\sqrt{x+h-5} + \sqrt{x-5}\right)}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h-5} + \sqrt{x-5}\right)} = \boxed{\frac{1}{2\sqrt{x-5}}}$$

② b) $f(x) = \dfrac{3}{\sqrt{x}} = \dfrac{3}{x^{1/2}} = 3x^{-1/2}$

$$f'(x) = -\frac{3}{2}x^{-3/2} = \frac{-3}{2x^{3/2}} = \frac{-3}{2\sqrt{x^3}}$$

Review

① b) $f(x)=\dfrac{2x-2}{x+3}$ $\qquad f(x+h)=\dfrac{2(x+h)-2}{(x+h)+3}=\dfrac{2x+2h-2}{x+h+3}$

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{2x+2h-2}{x+h+3}-\dfrac{2x-2}{x+3}}{h}$$

CD: $(x+3)(x+h+3)$

$$f'(x)=\lim_{h\to 0}\frac{(x+3)(2x+2h-2)-(2x-2)(x+h+3)}{h(x+3)(x+h+3)}$$

$$f'(x)=\lim_{h\to 0}\frac{\cancel{2x^2}\cancel{+2xh}\cancel{-2x}\cancel{+6x}+6h\cancel{-6}-(\cancel{2x^2}\cancel{+2xh}\cancel{+6x}\cancel{-2x}-2h\cancel{-6})}{h(x+3)(x+h+3)}$$

$$f'(x)=\lim_{h\to 0}\frac{8\cancel{h}}{\cancel{h}(x+3)(x+h+3)}=\boxed{\frac{8}{(x+3)^2}}$$

Review:

2 a) $f(x) = 3x^2 + 5x - 2$

$f'(x) = 6x + 5$

c) $f(x) = 2x^4 + \sqrt{x} = 2x^4 + x^{1/2}$

$f'(x) = 8x^3 + \frac{1}{2}x^{-1/2}$

$f'(x) = 8x^3 + \frac{1}{2x^{1/2}}$

d) $f(x) = \sqrt[3]{x^2} = x^{2/3}$

$f'(x) = \frac{2}{3}x^{-1/3}$

$f'(x) = \frac{2}{3x^{1/3}}$

Review:

③ a) $y = (3x^2 - 2)(4x+5)$

$y' = (6x)(4x+5) + 4(3x^2-2)$

$y' = 24x^2 + 30x + 12x^2 - 8$

$y' = 36x^2 + 30x - 8$

③ b) $g(x) = (x^2 - 5x + 2)(4x+1)$

$g'(x) = (2x-5)(4x+1) + 4(x^2-5x+2)$

$g'(x) = 8x^2 + 2x - 20x - 5 + 4x^2 - 20x + 8$

$g'(x) = 12x^2 - 38x + 3$

Review:

④ a) $f(x) = \frac{2x^2+3}{3x-2}$

$$f'(x) = \frac{4x(3x-2) - 3(2x^2+3)}{(3x-2)^2}$$

$$f'(x) = \frac{12x^2 - 8x - 6x^2 - 9}{(3x-2)^2}$$

$$f'(x) = \frac{6x^2-8x-9}{(3x-2)^2}$$

④ b) $y = \frac{\sqrt{x}}{3+x^2} = \frac{x^{1/2}}{3+x^2}$ $f(x)$ $g(x)$

$$y' = \frac{\frac{1}{2}x^{-1/2}(3+x^2) - (2x)(x^{1/2})}{(3+x^2)^2}$$

$$y' = \frac{\frac{(3+x^2)}{\cancel{2x^{1/2}}}\cancel{2x^{1/2}} - \frac{2x^{3/2}}{1}\cdot 2x^{1/2}}{(3+x^2)^2 \; 2x^{1/2}}$$

CD: $2x^{1/2}$

$$y' = \frac{3+x^2-4x^2}{2\sqrt{x}(3+x^2)^2} = \frac{3-3x^2}{2\sqrt{x}(3+x^2)^2}$$

Review

⑤ Find the equation of the tangent line to the curve $y=(x^2-3)^8$ at $x=2 \leftarrow x_1$

(i) Find y:

$y=(2^2-3)^8$

$y=(4-3)^8$

$y=1^8$

$y=1 \leftarrow y_1$

(ii) Find y'

$y=(x^2-3)^8$

$y'=8(x^2-3)^7(2x)$

$y'=16x(x^2-3)^7$

(iii) Find m $(y'(2))$

$y'=16(2)[(2)^2-3]^7$

$y'=32(1)$

$y'=32 \leftarrow m$

(iv) Find equation:

$y-y_1=m(x-x_1)$

$y-1=32(x-2)$

$y-1=32x-64$

$\boxed{y=32x-63}$ or $32x-y-63=0$

Review:

⑥ a) $f(x) = 3(2x^2-4)^4$

$f'(x) = 12(2x^2-4)^3(4x)$

$f'(x) = 48x(2x^2-4)^3$

b) $y = \dfrac{16}{\sqrt{x-1}} = \dfrac{16}{(x-1)^{1/2}} = 16(x-1)^{-1/2}$

$y' = -8(x-1)^{-3/2}(1)$

$y' = -8(x-1)^{-3/2}$

$y' = \dfrac{-8}{(x-1)^{3/2}}$

Review:

① a) $f(x) = \left(\frac{2x+1}{x-1}\right)^5 = \frac{(2x+1)^5}{(x-1)^5}$

$f'(x) = 5\left(\frac{2x+1}{x-1}\right)^4 \left[\frac{2(x-1) - 1(2x+1)}{(x-1)^2}\right]$

$f'(x) = 5\frac{(2x+1)^4}{(x-1)^4}\left[\frac{\cancel{2x} - 2 - \cancel{2x} - 1}{(x-1)^2}\right]$

$f'(x) = 5 \cdot \frac{(2x+1)^4}{(x-1)^4} \cdot \frac{-3}{(x-1)^2} = \boxed{\frac{-15(2x+1)^4}{(x-1)^6}}$

Review

⑦ b) $y=(x^2-1)^3(3x-2)^2$

$y'=3(x^2-1)^2(2x)(3x-2)^2+(x^2-1)^3(2)(3x-2)(3)$

$y'=6x(x^2-1)^2(3x-2)^2+6(x^2-1)^3(3x-2)$

$y'=6(x^2-1)^2(3x-2)[\overset{3x^2-2x\ +x^2-1}{x(3x-2)+(x^2-1)}]$

$y'=6(x^2-1)^2(3x-2)(4x^2-2x-1)$

Review

① c) $y = \dfrac{(2x+1)^2}{(x^4-x+1)^2}$ $\begin{matrix} f(x) \\ g(x) \end{matrix}$

$$\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{2(2x+1)^1(2)(x^4-x+1)^2 - (2x+1)^2(2)(x^4-x+1)^1(4x^3-1)}{(x^4-x+1)^4}$$

$$y' = \frac{4(2x+1)(x^4-x+1)^2 - 2(4x^3-1)(2x+1)^2(x^4-x+1)}{(x^4-x+1)^4}$$

$$y' = \frac{2(2x+1)(x^4-x+1)\Big[\overset{2x^4-2x+2}{2(x^4-x+1)} - \overset{(8x^4+4x^3-2x-1)}{(4x^3-1)(2x+1)}\Big]}{(x^4-x+1)^4}$$

$$y' = \frac{2(2x+1)\cancel{(x^4-x+1)}(-6x^4-4x^3+3)}{(x^4-x+1)^{\cancel{4}\,3}}$$

$$y' = \frac{-2(2x+1)(6x^4+4x^3-3)}{(x^4-x+1)^3}$$

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④ Find $\left.\frac{dy}{dt}\right]_{t=1}$

$y=\sqrt{1+r^2}$ $\qquad r=\frac{t+1}{2t+1}$

(i) Find $\frac{dy}{dr}$:

$y=(1+r^2)^{1/2}$

$\frac{dy}{dr}=\frac{1}{\cancel{2}}(1+r^2)^{-1/2}(\cancel{2}r)$

$\frac{dy}{dr}=\frac{r}{\sqrt{1+r^2}}$

(ii) Find $\frac{dr}{dt}$:

$r=\frac{t+1}{2t+1}$

$2t+1-2t-2$

$\frac{dr}{dt}=\frac{1(2t+1)-2(t+1)}{(2t+1)^2}$

$\frac{dr}{dt}=\frac{-1}{(2t+1)^2}$

(iii) Find r when $t=1$

$r=\frac{t+1}{2t+1}$

$r=\frac{1+1}{2(1)+1}$

$r=\frac{2}{3}$

(iv) Find $\left.\frac{dy}{dt}\right]_{t=1}$

$\left.\frac{dy}{dt}\right]_{t=1}=\underbrace{\left[\frac{dy}{dr}\right]}_{\text{Step 1}}\underbrace{\left[\frac{dr}{dt}\right]}_{\text{Step 2}}$

$\left.\frac{dy}{dt}\right]_{t=1}=\left[\frac{r}{\sqrt{1+r^2}}\right]\left[\frac{-1}{(2t+1)^2}\right]$

$\left.\frac{dy}{dt}\right]_{t=1}=\left[\frac{2/3}{\sqrt{1+(2/3)^2}}\right]\left[\frac{-1}{(2(1)+1)^2}\right]$

$\left.\frac{dy}{dt}\right]_{t=1}=\left[\frac{2/3}{\sqrt{9/9+4/9}}\right]\left[\frac{-1}{9}\right]$

$\left.\frac{dy}{dt}\right]_{t=1}=\left[\frac{2/3}{\sqrt{13/9}}\right]\left[\frac{-1}{9}\right]$

$\left.\frac{dy}{dt}\right]_{t=1}=\left[\frac{2}{\cancel{3}}\cdot\frac{\cancel{3}}{\sqrt{13}}\right]\left[\frac{-1}{9}\right]$

$\left.\frac{dy}{dt}\right]_{t=1}=\left[\frac{2}{\sqrt{13}}\right]\left[\frac{-1}{9}\right]$

$\left.\frac{dy}{dt}\right]_{t=1}=\frac{-2}{9\sqrt{13}}$

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⑥ a) $F(x) = x\sqrt{x^2+1}$

$F(x) = x(x^2+1)^{1/2}$

$F'(x) = 1(x^2+1)^{1/2} + x\left(\frac{1}{\cancel{2}}\right)(x^2+1)^{-1/2}(\cancel{2}x)$

$F'(x) = (x^2+1)^{1/2} + x^2(x^2+1)^{-1/2}$

x^2+1+x^2

$F'(x) = (x^2+1)^{-1/2}\left[(x^2+1)+x^2\right]$

$F'(x) = (x^2+1)^{-1/2}(2x^2+1)$

$F'(x) = \dfrac{2x^2+1}{\sqrt{x^2+1}}$

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⑥ e) $F(x) = \dfrac{x}{\sqrt{2x+3}} = x(2x+3)^{-1/2}$

$$F'(x) = 1(2x+3)^{-1/2} + x\left(-\frac{1}{2}\right)(2x+3)^{-3/2}(2)$$

$$F'(x) = (2x+3)^{-1/2} - x(2x+3)^{-3/2}$$

$$F'(x) = (2x+3)^{-3/2}\left[(2x+3) - x\right]$$

$$F'(x) = \frac{x+3}{(2x+3)^{3/2}} = \frac{x+3}{\sqrt{(2x+3)^3}}$$

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⑧ $y = \frac{1}{\sqrt{20-x^4}}$ @ $(2, \frac{1}{2})$ $x_1 = 2$ $y_1 = \frac{1}{2}$

(i) Find $\frac{dy}{dx}$:

$y = \frac{1}{(20-x^4)^{1/2}}$

$y = 1(20-x^4)^{-1/2}$

$\frac{dy}{dx} = -\frac{1}{2}(20-x^4)^{-3/2}(-4x^3)$

$\frac{dy}{dx} = \frac{2x^3}{(20-x^4)^{3/2}}$

$\frac{dy}{dx} = \frac{2x^3}{(\sqrt{20-x^4})^3}$

(ii) Find $\frac{dy}{dx}$ when $x=2$

$\frac{dy}{dx}\Big]_{x=2} = \frac{2(2)^3}{(\sqrt{20-(2)^4})^3}$

$\frac{dy}{dx}\Big]_{x=2} = \frac{16}{8}$

$\frac{dy}{dx}\Big]_{x=2} = 2$

$m = 2$

(iii) Find equation:

$y - y_1 = m(x - x_1)$

$y - \frac{1}{2} = 2(x - 2)$

$y - \frac{1}{2} = 2x - 4$

$y = 2x - 4 + \frac{1}{2}$

$y = 2x - \frac{8}{2} + \frac{1}{2}$

$\boxed{y = 2x - \frac{7}{2}}$ ✓

$2y = 4x - 7$

$\boxed{0 = 4x - 2y - 7}$

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(9) $F(x) = f(g(x))$ ← Composite Function

Given:

$g(2) = 4$

$g'(2) = 3$

$f'(4) = 5$

$f(2) = -1$

$f'(2) = 2$

find $F'(2)$

$F'(x) = f'(g(x)) \cdot g'(x)$

$F'(2) = f'(g(2)) \cdot g'(2)$

$F'(2) = f'(4) \cdot g'(2)$

$F'(2) = 5 \cdot 3$

$F'(2) = 15$