Warm Up

$$\lim_{x\to0}\frac{(x+2)^4-16}{x}$$

$$=\lim_{x\to0}\frac{\left[(x+2)^2-4\right]\left[(x+2)^2+4\right]}{x}$$

$$=\lim_{x\to0}\frac{\left[x+2-2\right]\left[x+2+2\right]\left[(x+2)^2+4\right]}{x}$$

$$=\lim_{x\to0}\frac{\overset{1}{\cancel{(x)}}(x+4)\left[(x+2)^2+4\right]}{\underset{1}{\cancel{x}}}$$

$$=(4)(8)$$

$$=32$$

$$\lim_{x\to0}\frac{\cancel{(1+x)}\frac{1}{\cancel{1+x}}-1(1+x)}{x(1+x)}$$

CD: $(1+x)\cdot 1 \cdot 1$

$(1+x)$

$$=\lim_{x\to0}\frac{1-1-x}{x(1+x)}$$

$$=\lim_{x\to0}\frac{\overset{-1}{\cancel{-x}}}{\underset{1}{\cancel{x}}(1+x)}$$

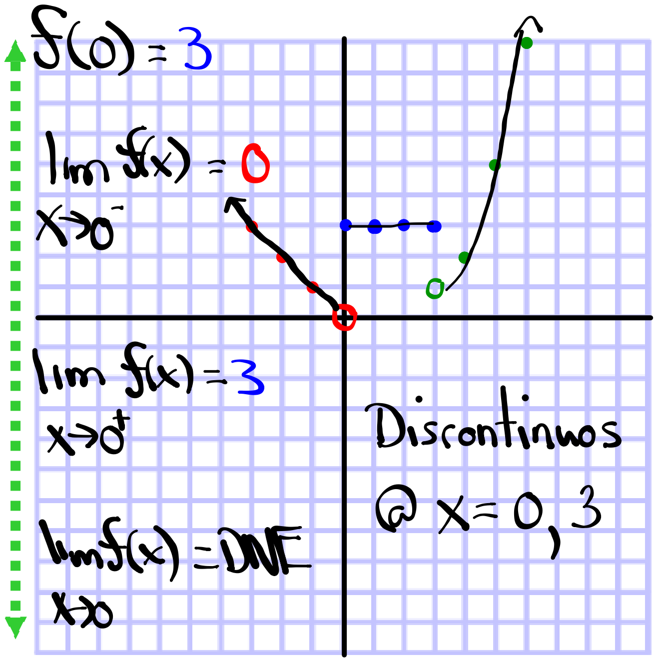
$$=\frac{-1}{1}$$

$$=-1$$

Questions From Homework

② $f(x) = \begin{cases} |x| & x<0 \\ 3 & 0 \le x \le 3 \\ (x-3)^2+1 & x>3 \end{cases}$

$>, < \rightarrow$ open
$\ge, \le, = \rightarrow$ closed



$|x|$

x	y
0	0
-1	1
-2	2
-3	3

3

x	y
0	3
1	3
2	3
3	3

$(x-3)^2+1$

x	y
3	1
4	2
5	5
6	10

Questions From Homework

④ d) $\lim\limits_{x\to 0} \dfrac{\dfrac{\cancel{4(2+x)^2}\,1}{\cancel{(2+x)^2}} - \dfrac{1}{\cancel{4}}\,\cancel{4}(2+x)^2}{x\;4(2+x)^2}$ CD: $4(2+x)^2$

$= \lim\limits_{x\to 0} \dfrac{4-(2+x)^2}{4x(2+x)^2}$ diff of squares

$= \lim\limits_{x\to 0} \dfrac{[2-(2+x)][2+(2+x)]}{4x(2+x)^2}$

$= \lim\limits_{x\to 0} \dfrac{[2-2-x][2+2+x]}{4x(2+x)^2}$

$= \lim\limits_{x\to 0} \dfrac{\overset{-1}{\cancel{(-x)}}(4+x)}{4\cancel{x}(2+x)^2}$

$= \dfrac{-4}{16}$

$= \dfrac{-1}{4}$

e) $\lim\limits_{x\to -4} \dfrac{x+4}{x^3+64}$

$= \lim\limits_{x\to -4} \dfrac{\cancel{(x+4)}}{\cancel{(x+4)}(x^2-4x+16)}$

$= \dfrac{1}{16+16+16}$

$= \dfrac{1}{48}$

Continuity

Definition

- We noticed in the preceding section that…
  - the limit of a function as x approaches a

  can often be found simply by…
  - calculating the value of the function at a.
- Functions with this property are called *continuous* at a:

1 Definition A function f is **continuous at a number a** if

$$\lim_{x \to a} f(x) = f(a)$$

- This definition implicitly requires three things if f is continuous at a:
  1. $f(a)$ is defined
     - That is, a is in the domain of f
  2. $f(x)$ has a limit as x approaches a
  3. This limit is actually equal to $f(a)$.

$\lim_{x \to a} f(x) = f(a)$

In English!

- Graph must be defined at that point $f(1) = 3$
- Limit from left and right must be equal $\lim_{x \to 1} f(x) = 1$
- Limit must be the same as the defined height of the function $f(1) \neq \lim_{x \to 1} f(x)$

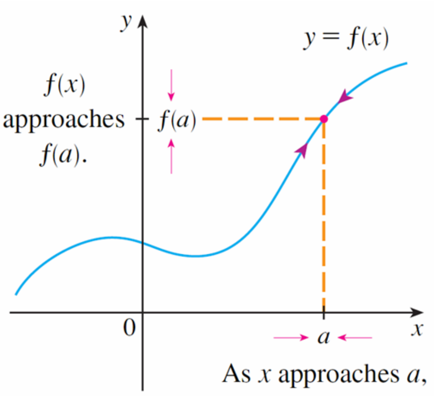


FIGURE 1

Let's simplify things...

A function whose graph has holes or breaks is considered discontinuous at these particular points.

If you have to lift your pencil from the page to sketch the graph, it is discontinuous anywhere you lift your pencil

Try this one...

$$f(x)=\begin{cases} 2-x^2 & if\ x<\underline{1} \\ 3 & if\ x=\underline{1} \\ 2x-1 & if\ \underline{1}<x\leq\underline{3} \\ (x-4)^2 & if\ x>\underline{3} \end{cases}$$

Possible discontinuity
@ $x=1$
$x=3$

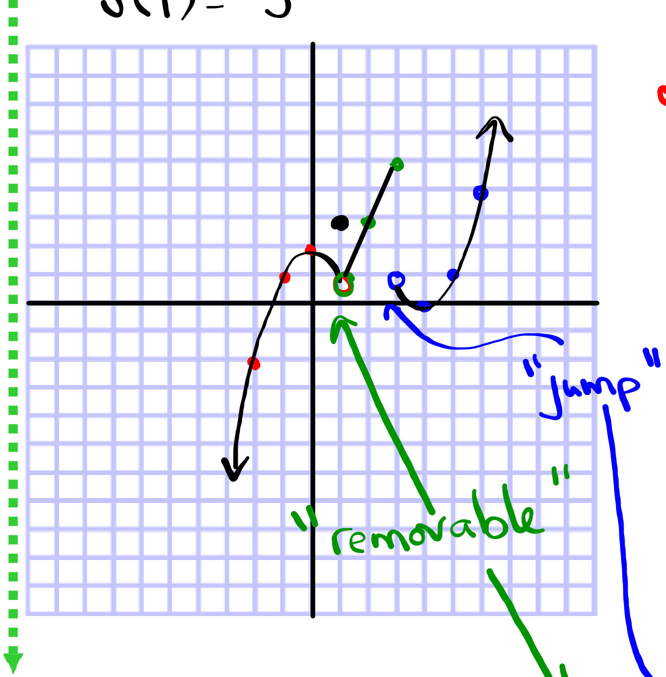
Evaluate:

$\lim_{x\to1} f(x) = 1$

$f(1) = 3$

$\lim_{x\to3} f(x) = \text{DNE}$

$f(3) = 5$



$2-x^2$

x	$f(x)$
1	1
0	2
-1	1
-2	-2
⋮	⋮

3

x	$f(x)$
1	3

$2x-1$

x	$f(x)$
1	1
2	3
3	5

$(x-4)^2$

x	$f(x)$
3	1
4	0
5	1
6	4
⋮	⋮

Discontinuous @ $x=1$ and $x=3$

$\lim_{x\to1} f(x) \neq f(1)$

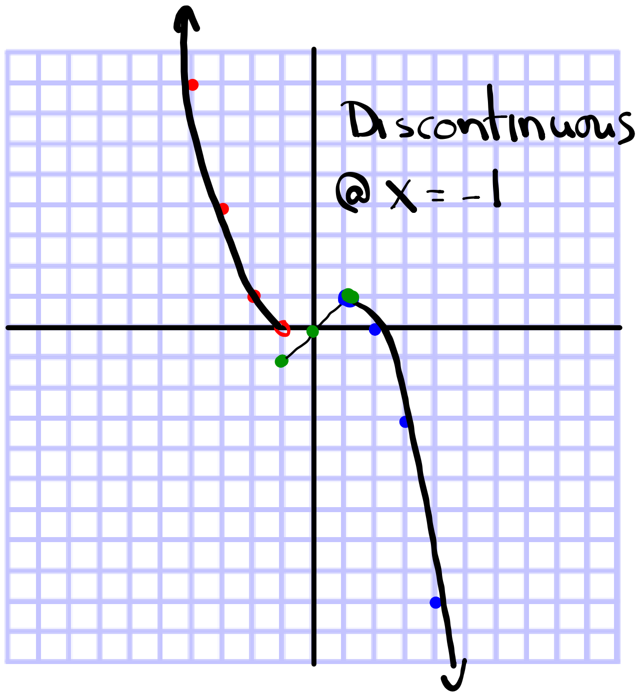
$\lim_{x\to3} f(x) = \text{DNE}$

Given the function

$$f(x) = \begin{cases} (x+1)^2 & x < -1 \\ x & -1 \le x \le 1 \\ 2x - x^2 & x > 1 \end{cases}$$

(a) Sketch $f(x)$.

(b) Check $f(x)$ for any points of discontinuity. Provide a mathematical reason to validate any point(s) where the function is discontinuous.



$(x+1)^2$

x	$f(x)$
-1	0
-2	1
-3	4
-4	9

x

x	$f(x)$
-1	-1
0	0
1	1

$2x - x^2$

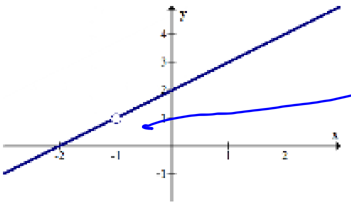
x	$f(x)$
1	1
2	0
3	-3
4	-8

A **removable discontinuity** occurs when the graph of a function has a hole.

For example, consider the following function:

$$f(x) = \frac{(x+2)(x+1)}{x+1}$$

Notice that the set of factors $x+1$ can be removed or canceled to get the function $f(x) = x+2$. The graph will then resemble $y = x+2$, except there will be a hole at $x = -1$ to account for the removed factor $x+1$. The reason for the hole is that although the original function can be simplified, -1 must be excluded from the domain of the function. Thus, graph the line $y = x+2$ as usual, but remove the point at $x = -1$:



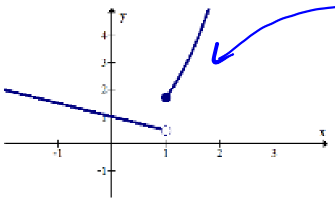
Discontinuous @ x = -1

① $f(-1) =$ undefined

Removable discontinuities can be "filled in" if you make the function a piecewise function and define a part of the function at the point where the hole is. In the example above, to make $f(x)$ continuous, you could redefine it as:

$$f(x) = \begin{cases} \frac{(x+2)(x+1)}{x+1}, & x \neq -1 \\ 1, & x = -1 \end{cases}$$

A **jump discontinuity** occurs when a function has two ends that don't meet, even if the hole is filled in at one of the ends. In order to satisfy the vertical line test and make sure the graph is truly that of a function, only one of the end points may be filled. Below is an example of a function with a jump discontinuity:

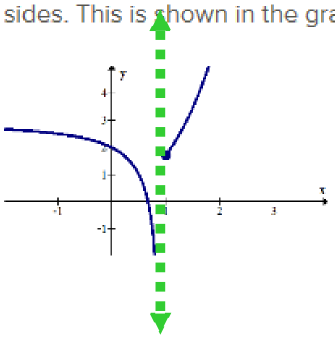


Discontinuous @ x = 1

① $f(1) = 2$

② $\lim_{x\to 1^-} f(x) = 0.5$ ish $\lim_{x\to 1^+} f(x) = 1.8$ ish

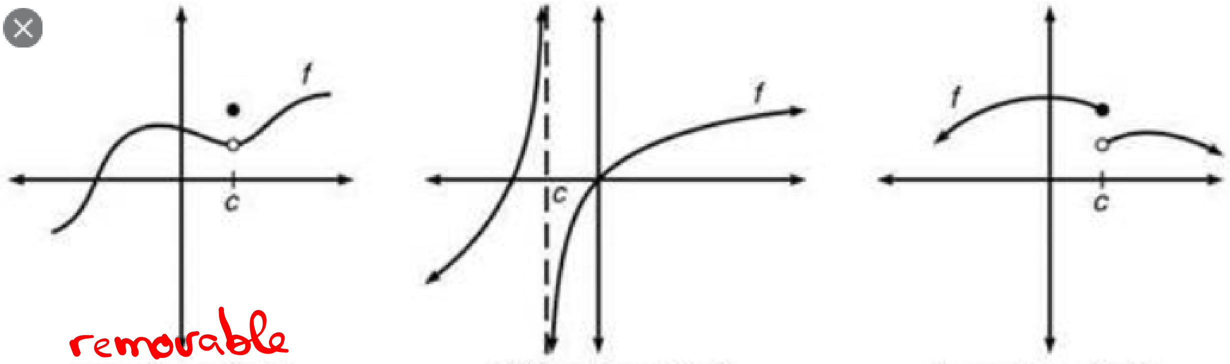
An **infinite discontinuity** occurs when a function has a vertical asymptote on one or both sides. This is shown in the graph of the function below at $x = 1$: (essential discontinuity)



Discontinuous @ x = 1

① $f(1) = 1.8$ ish

② $\lim_{x\to 1^-} f(x) = -\infty$ $\lim_{x\to 1^+} f(x) = 1.8$ ish



removable

point discontinuity
$\lim_{x \to c} f(x) \neq f(c)$
or
$f(c)$ does not exist

infinite discontinuity
(also called *essential discontinuity*)
$\lim_{x \to c} f(x) = \infty$ or $-\infty$

jump discontinuity
$\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$

$$\lim_{x \to c^-} f(x) = +\infty$$

$$\lim_{x \to c^+} f(x) = -\infty$$

https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-new/ab-1-10/v/types-of-discontinuities

. (6 points) Let $f(x) = \begin{cases} x^2-1, & \text{if } x \le \underline{0} \\ -3x-2, & \text{if } \underline{0} < x \le \underline{\underline{2}} \\ (4x^2-2x^3)/(x-2), & \text{if } x > \underline{\underline{2}} \end{cases}$

possible discontinuities @ $x=0$ or $x=2$

Find where f is continuous. Also, for each discontinuity, state what type of discontinuity it is (removable, jump, or essential).

@ $x=0$

defined height | left hand limit | right hand limit

① $f(x) = x^2-1$

$f(0) = (0)^2-1$

$f(0) = -1$

closed dot @ (0,-1)

② $\lim\limits_{x\to 0^-} x^2-1$

$= (0)^2-1$

$= -1$

③ $\lim\limits_{x\to 0^+} -3x-2$

$= -3(0)-2$

$= -2$

$$\boxed{\text{Since } \lim_{x\to 0} f(x) = \text{DNE we have jump discontinuity @ } x=0}$$

@ $x=2$

defined height | left hand limit | right hand limit

① $f(x) = -3x-2$

$f(2) = -3(2)-2$

$f(2) = -6-2$

$f(2) = -8$

closed dot @ (2,-8)

② $\lim\limits_{x\to 2^-} -3x-2$

$= -3(2)-2$

$= -6-2$

$= -8$

③ $\lim\limits_{x\to 2^+} \dfrac{4x^2-2x^3}{x-2}$

Indeterminant $\left(\frac{0}{0}\right)$

$= \lim\limits_{x\to 2^+} \dfrac{-2x^2(-2+x)}{(x-2)}$

$= \lim\limits_{x\to 2^+} \dfrac{-2x^2\cancel{(x-2)}}{\cancel{(x-2)}}$

$= -2(2)^2$

$= -8$

$$\boxed{\text{continuous @ } x=2}$$

Homework