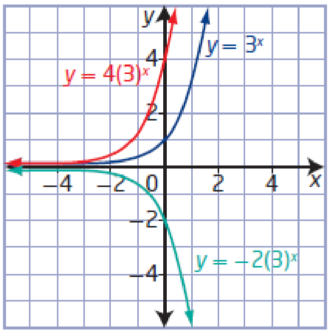
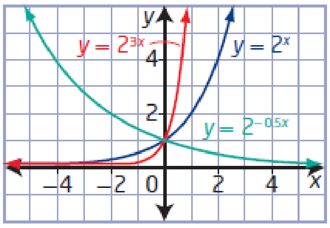
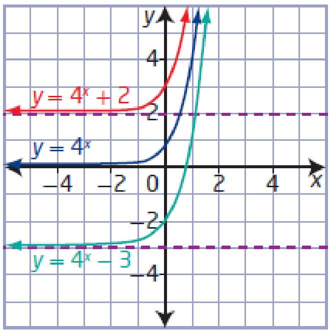
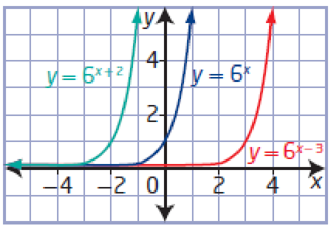
Transformations of Exponential Functions

Focus on...

- applying translations, stretches, and reflections to the graphs of exponential functions
- representing these transformations in the equations of exponential functions
- solving problems that involve exponential growth or decay

Link the Ideas

The graph of a function of the form $f(x) = a(c)^{b(x-h)} + k$ is obtained by applying transformations to the graph of the base function $y = c^x$, where $c > 0$.

Parameter	Transformation	Example
a	• Vertical stretch about the x-axis by a factor of $\lvert a \rvert$ • For $a < 0$, reflection in the x-axis • $(x, y) \rightarrow (x, ay)$	
b	• Horizontal stretch about the y-axis by a factor of $\frac{1}{\lvert b \rvert}$ • For $b < 0$, reflection in the y-axis • $(x, y) \rightarrow \left(\frac{x}{b}, y\right)$	
k	• Vertical translation up or down • $(x, y) \rightarrow (x, y + k)$	
h	• Horizontal translation left or right • $(x, y) \rightarrow (x + h, y)$	

Example 1

Apply Transformations to Sketch a Graph

Consider the base function $y = 3^x$. For each transformed function,

i) state the parameters and describe the corresponding transformations

ii) create a table to show what happens to the given points under each transformation

$y = 3^x$
$\left(-1, \frac{1}{3}\right)$
(0, 1)
(1, 3)
(2, 9)
(3, 27)

iii) sketch the graph of the base function and the transformed function

iv) describe the effects on the domain, range, equation of the horizontal asymptote, and intercepts

a) $y = 2(3)^{x-4}$

b) $y = -\frac{1}{2}(3)^{\frac{1}{5}x} - 5$

Solution

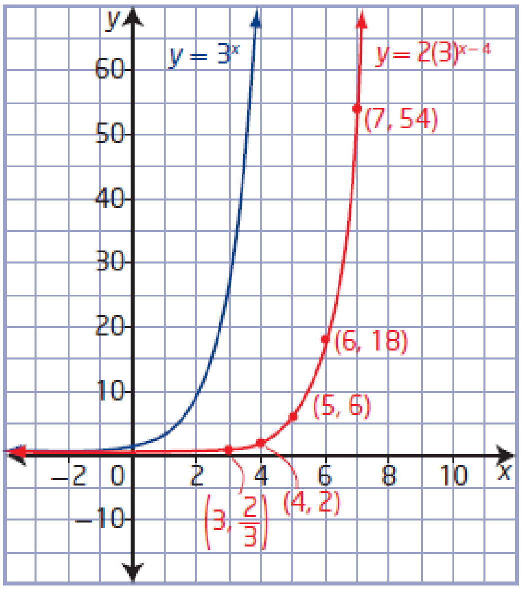
a) **i)** Compare the function $y = 2(3)^{x-4}$ to $y = a(c)^{b(x-h)} + k$ to determine the values of the parameters.

- $b = 1$ corresponds to no horizontal stretch.
- $a = 2$ corresponds to a vertical stretch of factor 2. Multiply the y-coordinates of the points in column 1 by 2.
- $h = 4$ corresponds to a translation of 4 units to the right. Add 4 to the x-coordinates of the points in column 2.
- $k = 0$ corresponds to no vertical translation.

ii) Add columns to the table representing the transformations.

$y = 3^x$	$y = 2(3)^{x-4}$
$\left(-1, \frac{1}{3}\right)$	$(3, 2/3)$
(0, 1)	(4, 2)
(1, 3)	(5, 6)
(2, 9)	(6, 18)
(3, 27)	(7, 54)

iii) To sketch the graph, plot the points from column 3 and draw a smooth curve through them.



iv) The domain remains the same: $\{x \mid x \in \mathrm{R}\}$.

The range also remains unchanged: $\{y \mid y > 0, y \in \mathrm{R}\}$.

The equation of the asymptote remains as $y = 0$.

There is still no x-intercept, but the y-intercept changes to $\frac{2}{81}$ or approximately 0.025.

b) $y = -\frac{1}{2}(3)^{\frac{1}{5}x} - 5$

i) state the parameters and describe the corresponding transformations
ii) create a table to show what happens to the given points under each transformation
iii) sketch the graph of the base function and the transformed function
iv) describe the effects on the domain, range, equation of the horizontal asymptote, and intercepts

b) $y = \left(-\frac{1}{2}\right)(3)^{\frac{1}{5}x} - 5$

c = base = 3

(i) $a = -\frac{1}{2}$ → vertical stretch by a factor of $\frac{1}{2}$ and a vertical reflection in the x-axis

$b = \frac{1}{5}$ → horizontal by a factor of 5

h = 0 → no horizontal translation

k = -5 → vertical translation 5 units down

(ii) $(x, y) \longrightarrow \left[5x, -\frac{1}{2}y - 5\right]$

$y = 3^x$

x	y
-2	1/9
-1	1/3
0	1
1	3
2	9

x	y
-10	-91/18 = -5.06
-5	-31/6 = -5.17
0	-11/2 = -5.5
5	-13/2 = -6.5
10	-19/2 = -9.5

$-\frac{1}{2}\left(\frac{1}{9}\right) - 5$

$-\frac{1}{18} - \frac{90}{18} = -\frac{91}{18}$

$-\frac{1}{2}\left(\frac{1}{3}\right) - 5$

$-\frac{1}{6} - \frac{30}{6} = \frac{-31}{6}$

$-\frac{1}{2}\left(\frac{1}{1}\right) - 5$

$-\frac{1}{2} - \frac{10}{2} = -\frac{11}{2}$

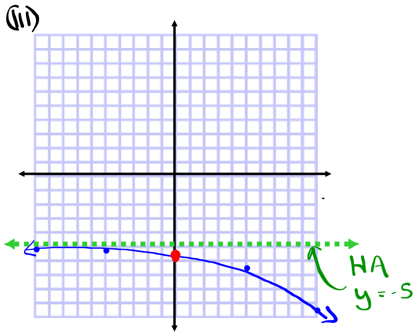
$-\frac{1}{2}\left(\frac{3}{1}\right) - 5$

$-\frac{3}{2} - \frac{10}{2} = -\frac{13}{2}$

$-\frac{1}{2}\left(\frac{9}{1}\right) - 5$

$-\frac{9}{2} - \frac{10}{2} = \frac{-19}{2}$

(iii)



(iv) D: $\{x \mid x \in \mathbb{R}\}$ or $(-\infty, \infty)$

R: $\{y \mid y < -5, y \in \mathbb{R}\}$ or $(-\infty, -5)$

HA: $y = -5$

x int (y = 0)

$y = -\frac{1}{2}(3)^{\frac{1}{5}x} - 5$

$0 = -\frac{1}{2}(3)^{\frac{1}{5}x} - 5$ (+5 both sides)

$\frac{5}{-1/2} = \frac{-\frac{1}{2}(3)^{\frac{1}{5}x}}{-\frac{1}{2}}$

$-10 = (3)^{\frac{1}{5}x}$

↳ $\frac{\log(-10)}{\log(3)}$ = error

No x-intercept

(Check graph)

y int (x = 0)

$y = -\frac{1}{2}(3)^{\frac{1}{5}x} - 5$

$y = -\frac{1}{2}(3)^{\frac{1}{5}(0)} - 5$

$y = -\frac{1}{2}(3)^{0} - 5$

$y = -\frac{1}{2}(1) - 5$

$y = \frac{-1}{2} - 5$

$y = -\frac{1}{2} - \frac{10}{2}$

$y = -\frac{11}{2} = -5.5$

(0, -5.5)

Your Turn

Transform the graph of $y = 4^x$ to sketch the graph of $y = 4^{-2(x+5)} - 3$. Describe the effects on the domain, range, equation of the horizontal asymptote, and intercepts.

$y = 1(4)^{-2(x+5)} - 3$ → $y = 4^x$

c = base = 4
a = 1
b = −2
h = −5
k = −3

x	y
−2	1/16
−1	1/4
0	1
1	4
2	16

x-int (y = 0)

$y = 4^{-2(x+5)} - 3$

$0 = 4^{-2(x+5)} - 3$

$3 = 4^{-2(x+5)}$

$\frac{\log(3)}{\log(4)} = 0.79248$ ← Base, ↑ exponent

$4^{0.79248} = 4^{-2(x+5)}$

$\frac{0.79248}{-2} = \frac{-2(x+5)}{-2}$

$-0.39624 - 5 = x + 5 - 5$

$-5.39624 = x$

$(-5.39624, 0)$

y-int (x = 0)

$y = 4^{-2(x+5)} - 3$

$y = 4^{-2(0+5)} - 3$

$y = 4^{-2(5)} - 3$

$y = 4^{-10} - 3$

$y = \left(\frac{1}{4}\right)^{10} - 3$

$y = \frac{1}{1048576} - \frac{3}{1}$

$y = \frac{1}{1048576} - \frac{3145728}{1048576}$

$y = \frac{-3145727}{1048576} = -2.\overline{9}$

$(0, -2.\overline{9})$

Homework

#1-7 and #10 on page 354