Understanding Logarithms

Focus on...

- demonstrating that a logarithmic function is the inverse of an exponential function
- sketching the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- determining the characteristics of the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- explaining the relationship between logarithms and exponents
- expressing a logarithmic function as an exponential function and vice versa
- evaluating logarithms using a variety of methods

Questions from Homework

5. Identify the following characteristics of the graph of each function.

i) the equation of the asymptote

ii) the domain and range

iii) the y-intercept, to one decimal place if necessary

iv) the x-intercept, to one decimal place if necessary

a) $y = -5 \log_3 (x + 3)$

b) $y = \log_6 (4(x + 9))$

c) $y = \log_5 (x + 3) - 2$

d) $y = -3 \log_2 (x + 1) - 6$

a) $y = -5\log_3(x+3)$ $c = 3$ (base)

$a = -5$ $b = 1$ $h = -3$ $k = 0$

(i) VA: $x = -3$

(ii) D: $\{x \mid x > -3, x \in R\}$ or $(-3, \infty)$

R: $\{y \mid y \in R\}$ or $(-\infty, \infty)$

(iii) y-int ($x=0$)

$y = -5\log_3(x+3)$

$y = -5\log_3(0+3)$

$y = -5\log_3(3)$ → $\frac{\log 3}{\log 3} = 1$

$y = -5(1)$

$y = -5$

(iv) x-int ($y=0$)

$y = -5\log_3(x+3)$

$\frac{0}{-5} = \frac{-5\log_3(x+3)}{-5}$

$0 = \log_3(x+3)$ (log)

$3^0 = x+3$ (exp)

$1 = x+3$

$-2 = x$

d) $y = -3\log_2(x+1) - 6$ $c = 2$ (base)

$a = -3$

$b = 1$

$h = -1$

$k = -6$

(i) VA: $x = -1$

(ii) D: $\{x \mid x > -1, x \in R\}$ or $(-1, \infty)$

R: $\{y \mid y \in R\}$

(iii) y-int ($x=0$)

$y = -3\log_2(x+1) - 6$

$y = -3\log_2(0+1) - 6$

$y = -3\log_2(1) - 6$ → $\frac{\log 1}{\log 2} = 0$

$y = -3(0) - 6$

$y = -6$

(iv) x-int ($y=0$)

$y = -3\log_2(x+1) - 6$

$0 = -3\log_2(x+1) - 6$

$\frac{6}{-3} = \frac{-3\log_2(x+1)}{-3}$

$-2 = \log_2(x+1)$ (log)

$2^{-2} = x+1$ (exp)

$\frac{1}{4} = x+1$

$-\frac{3}{4} = x$

or $x = -0.75$

Questions from Homework

11. Explain how the graph of $\frac{1}{3}(y + 2) = \log_6 (x - 4)$ can be generated by transforming the graph of $y = \log_6 x$.

3. $\frac{1}{3}(y+2) = \log_6(x-4)$ (multiply a + k)

$y + 2 = 3\log_6(x-4)$

$y = 3\log_6 1(x-4) - 2$

a = 3 → vertical stretch by a factor of 3

b = 1 → no horizontal stretch

h = 4 → translated 4 units right

k = -2 → " 2 units down.

x-int (y = 0)

$y = 3\log_6(x-4) - 2$

$0 = 3\log_6(x-4) - 2$

$\frac{2}{3} = \frac{3\log_6(x-4)}{3}$

$2/3 = \log_6(x-4)$ (log)

$6^{2/3} = x - 4$ (exp.)

$3.3 = x - 4$

$7.3 = x$

y-int (x = 0)

$y = 3\log_6(x-4) - 2$

$y = 3\log_6(0-4) - 2$

$y = 3\log_6(-4) - 2$ und.

No y int

General Properties of Logarithms:

If $c > 0$ and $c \neq 1$, then...

(i) $\log_c 1 = 0$

(ii) $\log_c c^x = x$

(iii) $c^{\log_c x} = x$

Did You Know?

The input value for a logarithm is called an argument. For example, in the expression $\log_6 1$, the argument is 1.

(i) $\log_5 1 = 0$ (ii) $\log_2 2^3 = 3$ (iii) $7^{\log_7 49} = 49$

$5^{\log_5 10} = 10$

③f) $\log_6 1 = 0$ ③a) $\log_6 6^4 = 4$ $5^{\log_5 10} = 10$

Product Law of Logarithms

The logarithm of a product of numbers can be expressed as the sum of the logarithms of the numbers.

$$\log_c MN = \log_c M + \log_c N$$

Proof

Let $\log_c M = x$ and $\log_c N = y$, where M, N, and c are positive real numbers with $c \neq 1$.

Write the equations in exponential form as $M = c^x$ and $N = c^y$:

$MN = (c^x)(c^y)$

$MN = c^{x+y}$ — Apply the product law of powers.

$\log_c MN = x + y$ — Write in logarithmic form.

$\log_c MN = \log_c M + \log_c N$ — Substitute for x and y.

Quotient Law of Logarithms

The logarithm of a quotient of numbers can be expressed as the difference of the logarithms of the dividend and the divisor.

$$\log_c \frac{M}{N} = \log_c M - \log_c N$$

Proof

Let $\log_c M = x$ and $\log_c N = y$, where M, N, and c are positive real numbers with $c \neq 1$.

Write the equations in exponential form as $M = c^x$ and $N = c^y$:

$$\frac{M}{N} = \frac{c^x}{c^y}$$

$$\frac{M}{N} = c^{x-y}$$ Apply the quotient law of powers.

$$\log_c \frac{M}{N} = x - y$$ Write in logarithmic form.

$$\log_c \frac{M}{N} = \log_c M - \log_c N$$ Substitute for x and y.

Power Law of Logarithms

The logarithm of a power of a number can be expressed as the exponent times the logarithm of the number.

$$\log_c M^P = P \log_c M$$

How could you prove the quotient law using the product law and the power law?

Proof

Let $\log_c M = x$, where M and c are positive real numbers with $c \neq 1$.

Write the equation in exponential form as $M = c^x$.
Let P be a real number.

$$M = c^x$$
$$M^P = (c^x)^P$$
$$M^P = c^{xP} \quad \text{Simplify the exponents.}$$
$$\log_c M^P = xP \quad \text{Write in logarithmic form.}$$
$$\log_c M^P = (\log_c M)P \quad \text{Substitute for } x.$$
$$\log_c M^P = P \log_c M$$

The laws of logarithms can be applied to logarithmic functions, expressions, and equations.

Product Law of Logarithms

The logarithm of a product of numbers can be expressed as the sum of the logarithms of the numbers.

$$\log_c MN = \log_c M + \log_c N$$

Ex: $\log 50 + \log 2 = \log(50 \cdot 2)$
$= \log 100$
$= 2$

Quotient Law of Logarithms

The logarithm of a quotient of numbers can be expressed as the difference of the logarithms of the dividend and the divisor.

$$\log_c \frac{M}{N} = \log_c M - \log_c N$$

Ex: $\log_6 36 - \log_6 4 = \log_6 \left(\frac{36}{4}\right)$
$= \log_6 9$
≈ 1.23

Power Law of Logarithms

The logarithm of a power of a number can be expressed as the exponent times the logarithm of the number.

$$\log_c M^P = P \log_c M$$

How could you prove the quotient law using the product law and the power law?

Ex. $\log_2 \sqrt{8}$

$= \log_2 8^{1/2}$

$= \frac{1}{2} \log_2 8$

$= \frac{1}{2}(3)$

$= \frac{3}{2}$

Homework

Finish Exercise 2

Example 1

Use the Laws of Logarithms to Expand Expressions

Write each expression in terms of individual logarithms of x, y, and z.

a) $\log_5 \frac{xy}{z}$

b) $\log_7 \sqrt[3]{x}$

c) $\log_6 \frac{1}{x^2}$

d) $\log \frac{x^3}{y\sqrt{z}}$

a) $\log_5 \frac{xy}{z} = \boxed{\log_5 x + \log_5 y - \log_5 z}$

b) $\log_7 \sqrt[3]{x} = \log_7 x^{\frac{1}{3}} = \boxed{\frac{1}{3}\log_7 x}$

c) $\log_6 \frac{1}{x^2} = \log_6 1 - \log_6 x^2$

$= 0 - 2\log_6 x$

$= \boxed{-2\log_6 x}$

d) $\log \frac{x^3}{y\sqrt{z}} = \log x^3 - (\log y + \log\sqrt{z})$

$= \log x^3 - \log y - \log z^{\frac{1}{2}}$

$= \boxed{3\log x - \log y - \frac{1}{2}\log z}$

Example 2

Use the Laws of Logarithms to Evaluate Expressions

Use the laws of logarithms to simplify and evaluate each expression.

a) $\log_6 8 + \log_6 9 - \log_6 2$

b) $\log_7 7\sqrt{7}$

c) $2\log_2 12 - \left(\log_2 6 + \frac{1}{3}\log_2 27\right)$

a) $\log_6 8 + \log_6 9 - \log_6 2$

$= \log_6\left(\frac{8 \cdot 9}{2}\right)$

$= \log_6 36$

$= 2$

b) $\log_7 7\sqrt{7}$

$= \log_7 7 + \log_7 7^{1/2}$

$= \log_7 7 + \frac{1}{2}\log_7 7$

$= 1 + \frac{1}{2}(1)$

$= 1 + \frac{1}{2}$

$= \frac{2}{2} + \frac{1}{2}$

$= 3/2$ or 1.5

c) $2\log_2 12 - \left(\log_2 6 + \frac{1}{3}\log_2 27\right)$

$2\log_2 12 - \log_2 6 - \frac{1}{3}\log_2 27$

$\log_2 12^2 - \log_2 6 - \log_2 27^{1/3}$

$\log_2 144 - \log_2 6 - \log_2 3$

$\log_2\left(\frac{144}{6 \cdot 3}\right)$

$\log_2 8$

3

Example 3

Use the Laws of Logarithms to Simplify Expressions

Write each expression as a single logarithm in simplest form. State the restrictions on the variable.

a) $\log_7 x^2 + \log_7 x - \dfrac{5 \log_7 x}{2}$

b) $\log_5 (2x - 2) - \log_5 (x^2 + 2x - 3)$

Key Ideas

- Let P be any real number, and M, N, and c be positive real numbers with $c \neq 1$. Then, the following laws of logarithms are valid.

Name	Law	Description
Product	$\log_c MN = \log_c M + \log_c N$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
Quotient	$\log_c \frac{M}{N} = \log_c M - \log_c N$	The logarithm of a quotient of numbers is the difference of the logarithms of the dividend and divisor.
Power	$\log_c M^P = P \log_c M$	The logarithm of a power of a number is the exponent times the logarithm of the number.

- Many quantities in science are measured using a logarithmic scale. Two commonly used logarithmic scales are the decibel scale and the pH scale.

Homework

Do I really understand??...

a) Express the following as a single logarithm... $2\log_2 3^2 + \log_2 6 - 3\log_2 3$

b) Evaluate the following... $\log_2 (32)^{\frac{1}{3}}$

c) Express the following as a single logarithm... $\frac{1}{2}\left[(\log_5 a + 2\log_5 b) - 3\log_5 c\right]$

d) Express as a single logarithm in simplest form...

$$\frac{3}{4}\left[12(\log_b x^2 - 2\log_b x) + 8\log_b \sqrt{x} - 4\log_b \frac{1}{x^7}\right]$$