Understanding Logarithms

Focus on...

- demonstrating that a logarithmic function is the inverse of an exponential function
- sketching the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- determining the characteristics of the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- explaining the relationship between logarithms and exponents
- expressing a logarithmic function as an exponential function and vice versa
- evaluating logarithms using a variety of methods

General Properties of Logarithms:

If $c > 0$ and $c \neq 1$, then...

(i) $\log_c 1 = 0$

(ii) $\log_c c^x = x$

(iii) $c^{\log_c x} = x$

Did You Know?

The input value for a logarithm is called an argument. For example, in the expression $\log_6 1$, the argument is 1.

(i) $\log_5 1 = 0$ (ii) $\log_2 2^3 = 3$ (iii) $7^{\log_7 49} = 49$

$5^{\log_5 10} = 10$

Product Law of Logarithms

The logarithm of a product of numbers can be expressed as the sum of the logarithms of the numbers.

$$\boxed{\log_c MN = \log_c M + \log_c N}$$

Proof

Let $\log_c M = x$ and $\log_c N = y$, where M, N, and c are positive real numbers with $c \neq 1$.

Write the equations in exponential form as $M = c^x$ and $N = c^y$:

$MN = (c^x)(c^y)$	
$MN = c^{x+y}$	Apply the product law of powers.
$\log_c MN = x + y$	Write in logarithmic form.
$\log_c MN = \log_c M + \log_c N$	Substitute for x and y.

Quotient Law of Logarithms

The logarithm of a quotient of numbers can be expressed as the difference of the logarithms of the dividend and the divisor.

$$\boxed{\log_c \frac{M}{N} = \log_c M - \log_c N}$$

Proof

Let $\log_c M = x$ and $\log_c N = y$, where M, N, and c are positive real numbers with $c \neq 1$.

Write the equations in exponential form as $M = c^x$ and $N = c^y$:

$$\frac{M}{N} = \frac{c^x}{c^y}$$

$$\frac{M}{N} = c^{x-y}$$ Apply the quotient law of powers.

$$\log_c \frac{M}{N} = x - y$$ Write in logarithmic form.

$$\log_c \frac{M}{N} = \log_c M - \log_c N$$ Substitute for x and y.

Power Law of Logarithms

The logarithm of a power of a number can be expressed as the exponent times the logarithm of the number.

$$\boxed{\log_c M^P = P \log_c M}$$

How could you prove the quotient law using the product law and the power law?

Proof

Let $\log_c M = x$, where M and c are positive real numbers with $c \neq 1$.

Write the equation in exponential form as $M = c^x$.
Let P be a real number.

$$M = c^x$$
$$M^P = (c^x)^P$$
$$M^P = c^{xP} \quad \text{Simplify the exponents.}$$
$$\log_c M^P = xP \quad \text{Write in logarithmic form.}$$
$$\log_c M^P = (\log_c M)P \quad \text{Substitute for } x.$$
$$\log_c M^P = P \log_c M$$

The laws of logarithms can be applied to logarithmic functions, expressions, and equations.

Product Law of Logarithms

The logarithm of a product of numbers can be expressed as the sum of the logarithms of the numbers.

$$\boxed{\log_c MN = \log_c M + \log_c N}$$

Ex: $\log 50 + \log 2 = \log(50 \cdot 2)$

$= \log 100$

$\boxed{= 2}$

Quotient Law of Logarithms

The logarithm of a quotient of numbers can be expressed as the difference of the logarithms of the dividend and the divisor.

$$\boxed{\log_c \frac{M}{N} = \log_c M - \log_c N}$$

Ex: $\log_6 36 - \log_6 4 = \log_6\left(\frac{36}{4}\right)$

$\boxed{= \log_6 9}$

≈ 1.23

Power Law of Logarithms

The logarithm of a power of a number can be expressed as the exponent times the logarithm of the number.

$$\boxed{\log_c M^P = P \log_c M}$$

How could you prove the quotient law using the product law and the power law?

Ex. $\log_2 \sqrt{8}$

$= \log_2 8^{1/2}$

$= \frac{1}{2}\boxed{\log_2 8}$

$= \frac{1}{2}(3)$

$\boxed{= \frac{3}{2}}$

Questions from Homework

Exercise 2

① h) $\log_r v = w$ (log form)

$\boxed{r^w = v}$ (exp form)

② h) $10^m = n$ (exp form)

$\log_{10} n = m$

$\boxed{\log n = m}$ (log form)

③ d) $\log_8 8^{17} = 17$

h) $\log_4 8 = 1.5 \longrightarrow$ Let $x = \log_4 8$

$\frac{\log 8}{\log 4} = 1.5$

$4^x = 8$

$(2^2)^x = (2^3)$

$\cancel{2}^{2x} = \cancel{2}^3$

$2x = 3$

$x = \frac{3}{2} = 1.5$

④ c) $\log_{10}(3x+5) = 2$ (log form)

$10^2 = 3x+5$ (exp form)

$100 = 3x+5$

$\frac{95}{3} = \frac{3x}{3}$

$\boxed{\frac{95}{3} = x}$

④ g) $\log_{\underset{\text{Base}}{2}}(\underbrace{\log_3 x}_{\text{ans}}) = \underset{\text{exp}}{4}$

$2^4 = \log_3 x$

$\underset{\text{exp}}{16} = \log_{\underset{\text{Base}}{3}} \underset{\text{ans}}{x}$

$3^{16} = x$

$\boxed{43\ 046\ 721 = x}$

④ e) $2^{1-x} = 3$ (exp form)

$\log_2 3 = 1-x$ (log form)

$x = 1 - \log_2 3$

Questions from Homework

Exercise 2

③ b) $\log_2 32 = 5$ g) $\log_3 (1/27) = -3$

$\frac{\log 32}{\log 2} = 5$ $\frac{\log(1/27)}{\log 3} = -3$

j) $\log_9 \sqrt{3} = 0.25$

$\frac{\log(\sqrt{3})}{\log 9} = 0.25$ or $\frac{1}{4}$

④ f) $\underset{\text{base}}{3}^{\overset{\text{exp}}{2x-1}} = \underset{\text{ans}}{5}$ (exp form)

$\log_3(5) = 2x-1$

$\frac{\log_3(5)+1}{2} = \frac{2x}{2}$

$\frac{\log_3(5)+1}{2} = x$

$\frac{1}{2}(\log_3(5)+1) = x$

h) $\underset{\text{Base}}{10}^{\overset{\text{exp}}{5^x}} = \underset{\text{ans}}{3}$ (Exp. form)

$\underbrace{\log_{10}(3)}_{\text{ans}} = \underset{\text{Base}}{5}^{\overset{\text{exp}}{x}}$ (Exp form)

$\log_5(\log_{10} 3) = x$

Example 1

Use the Laws of Logarithms to Expand Expressions

Write each expression in terms of individual logarithms of x, y, and z.

a) $\log_5 \frac{xy}{z}$

b) $\log_7 \sqrt[3]{x}$

c) $\log_6 \frac{1}{x^2}$

d) $\log \frac{x^3}{y\sqrt{z}}$

a) $\log_5 \frac{xy}{z}$

$\boxed{\log_5 x + \log_5 y - \log_5 z}$

b) $\log_7 \sqrt[3]{x}$

$\log_7 x^{\frac{1}{3}}$

$\boxed{\frac{1}{3}\log_7 x}$

c) $\log_6 \frac{1}{x^2}$

$\log_6 1 - \log_6 x^2$

$0 - 2\log_6 x$

$\boxed{-2\log_6 x}$

d) $\log \frac{x^3}{y\sqrt{z}}$

$\log x^3 - (\log y + \log \sqrt{z})$

$\log x^3 - \log y - \log z^{\frac{1}{2}}$

$\boxed{3\log x - \log y - \frac{1}{2}\log z}$

Example 2

Use the Laws of Logarithms to Evaluate Expressions

Use the laws of logarithms to simplify and evaluate each expression.

a) $\log_6 8 + \log_6 9 - \log_6 2$

b) $\log_7 7\sqrt{7}$

c) $2 \log_2 12 - \left(\log_2 6 + \frac{1}{3} \log_2 27\right)$

a) $\log_6 8 + \log_6 9 - \log_6 2$

$\log_6\left(\frac{8 \cdot 9}{2}\right)$

$\log_6(36)$

2

b) $\log_7 7\sqrt{7}$

$\log_7 7 + \log_7 \sqrt{7}$

$\log_7 7 + \log_7 7^{1/2}$

$1 + \frac{1}{2}$

$\frac{2}{2} + \frac{1}{2} = \frac{3}{2}$

c) $2\log_2 12 - (\log_2 6 + \frac{1}{3}\log_2 27)$

$\log_2 144 - (\log_2 6 + \log_2 3)$

$\log_2 144 - \log_2 6 - \log_2 3$

$\log_2\left(\frac{144}{6 \cdot 3}\right)$

$\log_2 8$

3

$\log_7 7\sqrt{7}$

$\log_7 7^1(7)^{1/2}$

$\log_7 7^{1+1/2}$

$\log_7 7^{3/2}$

$\frac{3}{2}$

Example 3

Use the Laws of Logarithms to Simplify Expressions

Write each expression as a single logarithm in simplest form. State the restrictions on the variable.

a) $\log_7 x^2 + \log_7 x - \dfrac{5 \log_7 x}{2}$

b) $\log_5 (2x - 2) - \log_5 (x^2 + 2x - 3)$

a) $\log_7 x^2 + \log_7 x - \frac{5}{2}\log_7 x$

$\log_7 x^2 + \log_7 x - \log_7 x^{5/2}$

$\log_7 \left(\frac{x^2 \cdot x}{x^{5/2}}\right)$

$\log_7 \left(\frac{x^3}{x^{5/2}}\right)$

$\log_7 x^{3-5/2}$

$\log_7 x^{6/2-5/2}$

$\log_7 x^{1/2}$

$\boxed{\frac{1}{2}\log_7 x}$

b) $\log_5(2x-2) - \log_5(x^2+2x-3)$

$\log_5 \left(\frac{2x-2}{x^2+2x-3}\right)$

common factor

trinomial

$-1 + 3 = 2$

$-1 \times 3 = -3$

$\log_5 \left[\frac{2\cancel{(x-1)}}{\cancel{(x-1)}(x+3)}\right]$

$\log_5 \left(\frac{2}{x+3}\right)$

For the original expression to be defined, both logarithmic terms must be defined.

$2x - 2 > 0$		$x^2 + 2x - 3 > 0$
$2x > 2$		$(x + 3)(x - 1) > 0$
$x > 1$	and	$x < -3$ or $x > 1$

What other methods could you have used to solve this quadratic inequality?

The conditions $x > 1$ and $x < -3$ or $x > 1$ are both satisfied when $x > 1$.

Hence, the variable x needs to be restricted to $x > 1$ for the original expression to be defined and then written as a single logarithm.

Therefore, $\log_5 (2x - 2) - \log_5 (x^2 + 2x - 3) = \log_5 \frac{2}{x + 3}$, $x > 1$.

Key Ideas

- Let P be any real number, and M, N, and c be positive real numbers with $c \neq 1$. Then, the following laws of logarithms are valid.

Name	Law	Description
Product	$\log_c MN = \log_c M + \log_c N$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
Quotient	$\log_c \frac{M}{N} = \log_c M - \log_c N$	The logarithm of a quotient of numbers is the difference of the logarithms of the dividend and divisor.
Power	$\log_c M^P = P \log_c M$	The logarithm of a power of a number is the exponent times the logarithm of the number.

- Many quantities in science are measured using a logarithmic scale. Two commonly used logarithmic scales are the decibel scale and the pH scale.

Homework
Finish Exercise 3

Questions from Homework

Exercise 3

③ e) $\frac{1}{2}\left[\log_5 x + 2\log_5 y - 3\log_5 z\right]$

$\frac{1}{2}\left[\log_5 x + \log_5 y^2 - \log_5 z^3\right]$

$\frac{1}{2}\log_5\left(\frac{xy^2}{z^3}\right)$

$\log_5\left(\frac{xy^2}{z^3}\right)^{1/2}$

$\log_5\sqrt{\frac{xy^2}{z^3}}$

$\log_5 y\sqrt{\frac{x}{z^3}}$

Do I really understand??...

a) Express the following as a single logarithm... $2\log_2 3^2 + \log_2 6 - 3\log_2 3$

b) Evaluate the following... $\log_2 (32)^{\frac{1}{3}}$

c) Express the following as a single logarithm... $\frac{1}{2}\left[(\log_5 a + 2\log_5 b) - 3\log_5 c\right]$

d) Express as a single logarithm in simplest form...

$$\frac{3}{4}\left[12\left(\log_b x^2 - 2\log_b x\right) + 8\log_b \sqrt{x} - 4\log_b \frac{1}{x^7}\right]$$