Understanding Logarithms

Focus on...

- demonstrating that a logarithmic function is the inverse of an exponential function
- sketching the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- determining the characteristics of the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- explaining the relationship between logarithms and exponents
- expressing a logarithmic function as an exponential function and vice versa
- evaluating logarithms using a variety of methods

Ex: $\log_5\left(\frac{x^2y^3}{w\sqrt{z}}\right)$

$\log_5 x^2y^3 - \log_5 w\sqrt{z}$

$\log_5 x^2 + \log_5 y^3 - (\log_5 w + \log_5 \sqrt{z})$

$\log_5 x^2 + \log_5 y^3 - \log_5 w - \log_5 \sqrt{z}$

$\log_5 x^2 + \log_5 y^3 - \log_5 w - \log_5 z^{\frac{1}{2}}$

$2\log_5 x + 3\log_5 y - \log_5 w - \frac{1}{2}\log_5 z$

Ex: 2 $5\log_3 x - \frac{1}{2}\log_3 x^4 + \log_3 x$

$\log_3 x^5 - \log_3 (x^4)^{\frac{1}{2}} + \log_3 x$

$\log_3 x^5 - \log_3 x^2 + \log_3 x$

$\log_3\left(\frac{x^5 \cdot x}{x^2}\right)$

$\log_3\left(\frac{x^6}{x^2}\right)$

$\log_3 x^4$

$4\log_3 x$

General Properties of Logarithms:

If $c > 0$ and $c \neq 1$, then...

(i) $\log_c 1 = 0$

(ii) $\log_c c^x = x$

(iii) $c^{\log_c x} = x$

Did You Know?

The input value for a logarithm is called an argument. For example, in the expression $\log_6 1$, the argument is 1.

(i) $\log_5 1 = 0$ (ii) $\log_2 2^3 = 3$ (iii) $7^{\log_7 49} = 49$

$5^{\log_5 10} = 10$

Product Law of Logarithms

The logarithm of a product of numbers can be expressed as the sum of the logarithms of the numbers.

$$\boxed{\log_c MN = \log_c M + \log_c N}$$

Proof

Let $\log_c M = x$ and $\log_c N = y$, where M, N, and c are positive real numbers with $c \neq 1$.

Write the equations in exponential form as $M = c^x$ and $N = c^y$:

$$MN = (c^x)(c^y)$$

$$MN = c^{x+y} \qquad \text{Apply the product law of powers.}$$

$$\log_c MN = x + y \qquad \text{Write in logarithmic form.}$$

$$\log_c MN = \log_c M + \log_c N \qquad \text{Substitute for } x \text{ and } y.$$

Quotient Law of Logarithms

The logarithm of a quotient of numbers can be expressed as the difference of the logarithms of the dividend and the divisor.

$$\boxed{\log_c \frac{M}{N} = \log_c M - \log_c N}$$

Proof

Let $\log_c M = x$ and $\log_c N = y$, where M, N, and c are positive real numbers with $c \neq 1$.

Write the equations in exponential form as $M = c^x$ and $N = c^y$:

$$\frac{M}{N} = \frac{c^x}{c^y}$$

$$\frac{M}{N} = c^{x-y}$$ Apply the quotient law of powers.

$$\log_c \frac{M}{N} = x - y$$ Write in logarithmic form.

$$\log_c \frac{M}{N} = \log_c M - \log_c N$$ Substitute for x and y.

Power Law of Logarithms

The logarithm of a power of a number can be expressed as the exponent times the logarithm of the number.

$\boxed{\log_c M^P = P \log_c M}$

How could you prove the quotient law using the product law and the power law?

Proof

Let $\log_c M = x$, where M and c are positive real numbers with $c \neq 1$.

Write the equation in exponential form as $M = c^x$.
Let P be a real number.

$$M = c^x$$
$$M^P = (c^x)^P$$
$$M^P = c^{xP} \quad \text{Simplify the exponents.}$$
$$\log_c M^P = xP \quad \text{Write in logarithmic form.}$$
$$\log_c M^P = (\log_c M)P \quad \text{Substitute for } x.$$
$$\log_c M^P = P \log_c M$$

The laws of logarithms can be applied to logarithmic functions, expressions, and equations.

Product Law of Logarithms

The logarithm of a product of numbers can be expressed as the sum of the logarithms of the numbers.

$$\log_c MN = \log_c M + \log_c N$$

Ex: $\log 50 + \log 2 = \log(50 \cdot 2)$

$= \log 100$

$\boxed{= 2}$

Quotient Law of Logarithms

The logarithm of a quotient of numbers can be expressed as the difference of the logarithms of the dividend and the divisor.

$$\log_c \frac{M}{N} = \log_c M - \log_c N$$

Ex: $\log_6 36 - \log_6 4 = \log_6\left(\frac{36}{4}\right)$

$\boxed{= \log_6 9}$

≈ 1.23

Power Law of Logarithms

The logarithm of a power of a number can be expressed as the exponent times the logarithm of the number.

$$\log_c M^P = P \log_c M$$

How could you prove the quotient law using the product law and the power law?

Ex. $\log_2 \sqrt{8}$

$= \log_2 8^{1/2}$

$= \frac{1}{2}\boxed{\log_2 8}$

$= \frac{1}{2}(3)$

$\boxed{= \frac{3}{2}}$

Express as a single logarithm:

① $\log_2 x + \log_2 y - \log_2 z + \log_2 P$

$\log_2\left(\frac{xyP}{z}\right)$

② $2\log_3 x - \frac{1}{2}\log_3 y - 4\log_3 z$

$\log_3 (x)^2 - \log_3 (y)^{\frac{1}{2}} - \log_3 (z)^4$

$\log_3\left(\frac{x^2}{y^{\frac{1}{2}} z^4}\right)$

$\log_3\left(\frac{x^2}{\sqrt{y}\, z^4}\right)$

Questions from Homework

Exercise 2

① h) $\log_r v = w$ (log form)

$\boxed{r^w = v}$ (exp form)

② h) $10^m = n$ (exp form)

$\log_{10} n = m$

$\boxed{\log n = m}$ (log form)

③ d) $\log_8 8^{17} = 17$

h) $\log_4 8 = 1.5$ $\longrightarrow$ Let $x = \log_4 8$

$\frac{\log 8}{\log 4} = 1.5$

$4^x = 8$

$(2^2)^x = (2^3)$

$\cancel{2}^{2x} = \cancel{2}^3$

$2x = 3$

$x = \frac{3}{2} = 1.5$

④ c) $\log_{10}(3x+5) = 2$ (log form)

$10^2 = 3x+5$ (exp form)

$100 = 3x+5$

$\frac{95}{3} = \frac{3x}{3}$

$\boxed{\frac{95}{3} = x}$

④ g) $\log_2(\underbrace{\log_3 x}_{\text{ans}}) = 4$

$\uparrow$ Base $\uparrow$ exp

$2^4 = \log_3 x$

$16 = \log_3 x$

exp, Base, ans

$3^{16} = x$

$\boxed{43\ 046\ 721 = x}$

④ e) $2^{1-x} = 3$ (exp form)

$\log_2 3 = 1-x$ (log form)

$x = 1 - \log_2 3$

Questions from Homework

Exercise 2

3 b) $\log_2 32 = 5$ g) $\log_3\left(\frac{1}{27}\right) = -3$

$\frac{\log 32}{\log 2} = 5$ $\frac{\log\left(\frac{1}{27}\right)}{\log 3} = -3$

j) $\log_9 \sqrt{3} = 0.25$

$\frac{\log(\sqrt{3})}{\log 9} = 0.25$ or $\frac{1}{4}$

4 f) $\underset{\text{base}}{3}^{2x-1 \leftarrow \text{exp}} = \underset{\text{ans}}{5}$ (exp form)

$\log_3(5) = 2x - 1$

$\frac{\log_3(5) + 1}{2} = \frac{2x}{2}$

$\frac{\log_3(5) + 1}{2} = x$

$\frac{1}{2}\left(\log_3(5) + 1\right) = x$

h) $\underset{\text{Base}}{10}^{5^x \leftarrow \text{exp}} = \underset{\text{ans}}{3}$ (Exp. form)

$\underbrace{\log_{10}(3)}_{\text{ans}} = \underset{\text{Base}}{5}^{x \leftarrow \text{exp}}$ (Exp form)

$\log_5(\log_{10} 3) = x$

Example 1

Use the Laws of Logarithms to Expand Expressions

Write each expression in terms of individual logarithms of x, y, and z.

a) $\log_5 \frac{xy}{z}$

b) $\log_7 \sqrt[3]{x}$

c) $\log_6 \frac{1}{x^2}$

d) $\log \frac{x^3}{y\sqrt{z}}$

a) $\log_5 \frac{xy}{z}$

$\boxed{\log_5 x + \log_5 y - \log_5 z}$

b) $\log_7 \sqrt[3]{x}$

$\log_7 x^{\frac{1}{3}}$

$\boxed{\frac{1}{3}\log_7 x}$

c) $\log_6 \frac{1}{x^2}$

$\log_6 1 - \log_6 x^2$

$0 - 2\log_6 x$

$\boxed{-2\log_6 x}$

d) $\log \frac{x^3}{y\sqrt{z}}$

$\log x^3 - (\log y + \log\sqrt{z})$

$\log x^3 - \log y - \log z^{\frac{1}{2}}$

$\boxed{3\log x - \log y - \frac{1}{2}\log z}$

Example 2

Use the Laws of Logarithms to Evaluate Expressions

Use the laws of logarithms to simplify and evaluate each expression.

a) $\log_6 8 + \log_6 9 - \log_6 2$

b) $\log_7 7\sqrt{7}$

c) $2 \log_2 12 - \left(\log_2 6 + \frac{1}{3} \log_2 27\right)$

a) $\log_6 8 + \log_6 9 - \log_6 2$

$\log_6\left(\frac{8 \cdot 9}{2}\right)$

$\log_6(36)$

2

b) $\log_7 7\sqrt{7}$

$\log_7 7 + \log_7 \sqrt{7}$

$\log_7 7 + \log_7 7^{1/2}$

$1 + \frac{1}{2}$

$\frac{2}{2} + \frac{1}{2} = \frac{3}{2}$

c) $2\log_2 12^2 - \left(\log_2 6 + \frac{1}{3}\log_2 27\right)$

$\log_2 144 - \left(\log_2 6 + \log_2 3\right)$

$\log_2 144 - \log_2 6 - \log_2 3$

$\log_2\left(\frac{144}{6 \cdot 3}\right)$

$\log_2 8$

3

$\log_7 7\sqrt{7}$

$\log_7 7^1(7)^{1/2}$

$\log_7 7^{1+1/2}$

$\log_7 7^{3/2}$

$\frac{3}{2}$

Example 3

Use the Laws of Logarithms to Simplify Expressions

Write each expression as a single logarithm in simplest form. State the restrictions on the variable.

a) $\log_7 x^2 + \log_7 x - \frac{5 \log_7 x}{2}$

b) $\log_5 (2x - 2) - \log_5 (x^2 + 2x - 3)$

2 terms

a) $\log_7 x^2 + \log_7 x - \frac{5}{2}\log_7 x$

$\log_7 x^2 + \log_7 x - \log_7 x^{5/2}$

$\log_7 \left(\frac{x^2 \cdot x}{x^{5/2}}\right)$

$\log_7 \left(\frac{x^3}{x^{5/2}}\right)$ ← $2+1 = 3$

$\log_7 x^{1/2}$ ← $3 - \frac{5}{2}$, $\frac{6}{2} - \frac{5}{2}$, $\frac{1}{2}$

$\frac{1}{2}\log_7 x$

b) $\log_5(2x-2) - \log_5(x^2+2x-3)$

$\log_5\left(\frac{2x-2}{x^2+2x-3}\right)$ ← common factor, trinomial

$\log_5\left[\frac{2\cancel{(x-1)}}{\cancel{(x-1)}(x+3)}\right]$ $-1 + 3 = 2$, $-1 \times 3 = -3$

$\log_5\left(\frac{2}{x+3}\right)$, $x > 1$

For the original expression to be defined, both logarithmic terms must be defined.

$2x - 2 > 0$ $\quad$ $x^2 + 2x - 3 > 0$

$2x > 2$ $\quad$ $(x + 3)(x - 1) > 0$

$x > 1$ $\quad$ and $\quad$ $x < -3$ or $x > 1$

What other methods could you have used to solve this quadratic inequality?

The conditions $x > 1$ and $x < -3$ or $x > 1$ are both satisfied when $x > 1$.

Hence, the variable x needs to be restricted to $x > 1$ for the original expression to be defined and then written as a single logarithm.

Therefore, $\log_5 (2x - 2) - \log_5 (x^2 + 2x - 3) = \log_5 \frac{2}{x + 3}$, $x > 1$.

Key Ideas

- Let P be any real number, and M, N, and c be positive real numbers with $c \neq 1$. Then, the following laws of logarithms are valid.

Name	Law	Description
Product	$\log_c MN = \log_c M + \log_c N$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
Quotient	$\log_c \frac{M}{N} = \log_c M - \log_c N$	The logarithm of a quotient of numbers is the difference of the logarithms of the dividend and divisor.
Power	$\log_c M^P = P \log_c M$	The logarithm of a power of a number is the exponent times the logarithm of the number.

- Many quantities in science are measured using a logarithmic scale. Two commonly used logarithmic scales are the decibel scale and the pH scale.

Homework
Finish Exercise 3

Exercise 2

④ g) $\log_2(\underbrace{\log_3 x}_{\text{ans}}) = 4$

Base ans exp

$2^4 = \log_3 x$

$16 = \log_3 x$

exp Base ans

$3^{16} = x$

$43\,046\,721 = x$

④ e) $2^{1-x} = 3$

Base exp ans

$\log_2(3) = 1 - x$

$x = 1 - \log_2(3)$

Exercise 3:

1) i) $\log_{10} \frac{x^3 y^4}{z^6}$

$\log x^3 + \log y^4 - \log z^6$

$3\log x + 4\log y - 6\log z$

j) $\log_{10} \frac{a^2}{b^4\sqrt{c}}$

$\log a^2 - (\log b^4 + \log c^{1/2})$

$\log a^2 - \log b^4 - \log c^{1/2}$

$2\log a - 4\log b - \frac{1}{2}\log c$

③ d) $4\log_2 x - \frac{1}{3}\log_2(x^2+1) + \log_2(x-1)$

$\log_2 x^4 - \log_2(x^2+1)^{1/3} + \log_2(x-1)$

$\log_2\left[\frac{x^4(x-1)}{(x^2+1)^{1/3}}\right]$

or

$\log_2\left[\frac{x^4(x-1)}{\sqrt[3]{x^2+1}}\right]$

③ c) $\log_5(x^2-1) - \log_5(x-1)$

$\log_5\left[\frac{(x^2-1)}{x-1}\right]$

difference of squares

$\log_5\left[\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}\right]$

$\log_5(x+1)$

Questions from Homework

Exercise 3

③ e) $\frac{1}{2}\left[\log_5 x + 2\log_5 y - 3\log_5 z\right]$

$\frac{1}{2}\left[\log_5 x + \log_5 y^2 - \log_5 z^3\right]$

$\frac{1}{2}\log_5\left(\frac{xy^2}{z^3}\right)$

$\log_5\left(\frac{xy^2}{z^3}\right)^{1/2}$

$\log_5\sqrt{\frac{xy^2}{z^3}}$

$\log_5 y\sqrt{\frac{x}{z^3}}$

Do I really understand??...

a) Express the following as a single logarithm... $2\log_2 3^2 + \log_2 6 - 3\log_2 3$

b) Evaluate the following... $\log_2 (32)^{\frac{1}{3}}$

c) Express the following as a single logarithm... $\frac{1}{2}\left[(\log_5 a + 2\log_5 b) - 3\log_5 c\right]$

d) Express as a single logarithm in simplest form...

$$\frac{3}{4}\left[12\left(\log_b x^2 - 2\log_b x\right) + 8\log_b \sqrt{x} - 4\log_b \frac{1}{x^7}\right]$$

Quiz:

Exercise 2

Exercise 3

Assignment → x and y-intercepts