

## Inverse of a Relation

An inverse function is a second function which undoes the work of the first one.

### 1. Introduction

Suppose we have a function  $f$  that takes  $x$  to  $y$ , so that

$$f(x) = y.$$

An inverse function, which we call  $f^{-1}$ , is another function that takes  $y$  back to  $x$ . So

$$f^{-1}(y) = x.$$

For  $f^{-1}$  to be an inverse of  $f$ , this needs to work for every  $x$  that  $f$  acts upon.

## Inverse of a Relation

The inverse of a relation is found by interchanging the  $x$ -coordinates and  $y$ -coordinates of the ordered pairs of the relation. In other words, for every ordered pair  $(x, y)$  of a relation, there is an ordered pair  $(y, x)$  on the inverse of the relation. This means that the graphs of a relation and its inverse are reflections of each other in the line  $y = x$ .

$$(x, y) \rightarrow (y, x)$$

In plain English....the  $x$  and  $y$  coordinates will just switch places

$$(-4, 2) \rightarrow (2, -4)$$

The inverse of a function  $y = f(x)$  may be written in the form  $x = f(y)$ . The inverse of a function is not necessarily a function. When the inverse of  $f$  is itself a function, it is denoted as  $f^{-1}$  and read as " $f$  inverse." When the inverse of a function is not a function, it may be possible to restrict the domain to obtain an inverse function for a portion of the original function.

The inverse of a function reverses the processes represented by that function. Functions  $f(x)$  and  $g(x)$  are inverses of each other if the operations of  $f(x)$  reverse all the operations of  $g(x)$  in the opposite order and the operations of  $g(x)$  reverse all the operations of  $f(x)$  in the opposite order.

For example,  $f(x) = 2x + 1$  multiplies the input value by 2 and then adds 1. The inverse function subtracts 1 from the input value and then divides by 2. The inverse function is  $f^{-1}(x) = \frac{x-1}{2}$ .

function

$$f(x) = 2x + 1$$

$$f(1) = 2(1) + 1$$

$$f(1) = 2 + 1$$

$$f(1) = 3$$

$$(1, 3)$$

Inverse function

$$f^{-1}(x) = \frac{x-1}{2}$$

$$f^{-1}(3) = \frac{3-1}{2} = \frac{2}{2} = 1$$

$$(3, 1)$$

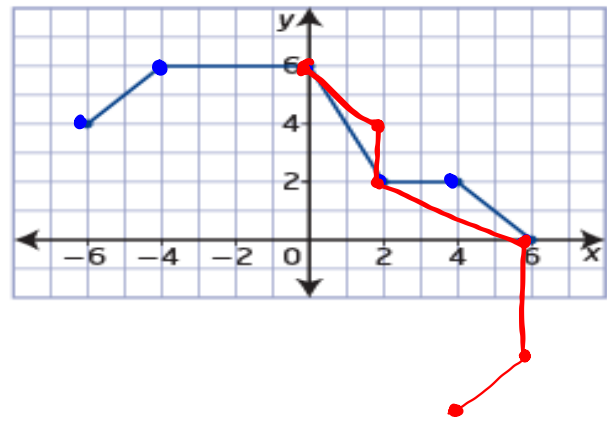
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### Example 1

#### Graph an Inverse

Consider the graph of the relation shown.

- Sketch the graph of the inverse relation.
- State the domain and range of the relation and its inverse.
- Determine whether the relation and its inverse are functions.

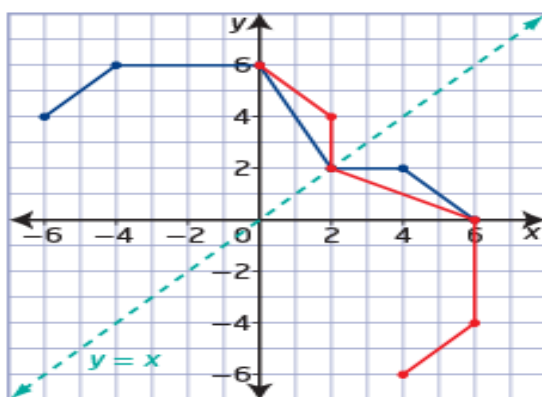


#### Solution

- To graph the inverse relation, interchange the  $x$ -coordinates and  $y$ -coordinates of key points on the graph of the relation.

Points on the Relation	Points on the Inverse Relation
$(-6, 4)$	$(4, -6)$
$(-4, 6)$	$(6, -4)$
$(0, 6)$	$(6, 0)$
$(2, 2)$	$(2, 2)$
$(4, 2)$	$(2, 4)$
$(6, 0)$	$(0, 6)$

\* Invariant



The graphs are reflections of each other in the line  $y = x$ . The points on the graph of the relation are related to the points on the graph of the inverse relation by the mapping  $(x, y) \rightarrow (y, x)$ .

What points are invariant after a reflection in the line  $y = x$ ?

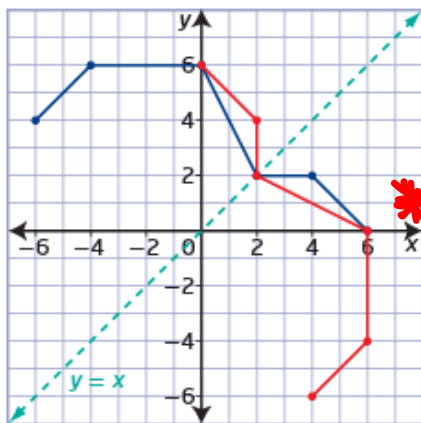
↳ any point where  $y = x$

ex.  $(2, 2)$

$(5, 5)$

$(-10, -10)$

b) State the domain and range of the relation and its inverse.



	$[-6, 6]$	$[0, 6]$
	Domain	Range
● Relation	$\{x \mid -6 \leq x \leq 6, x \in \mathbb{R}\}$	$\{y \mid 0 \leq y \leq 6, y \in \mathbb{R}\}$
● Inverse Relation	$\{x \mid 0 \leq x \leq 6, x \in \mathbb{R}\}$	$\{y \mid -6 \leq y \leq 6, y \in \mathbb{R}\}$

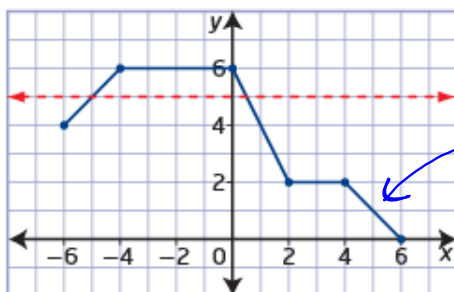
$[0, 6]$

$[-6, 6]$

The domain of the relation becomes the range of the inverse relation and the range of the relation becomes the domain of the inverse relation.

In plain English....the x and y coordinates will just switch places

c) Determine whether the relation and its inverse are functions.

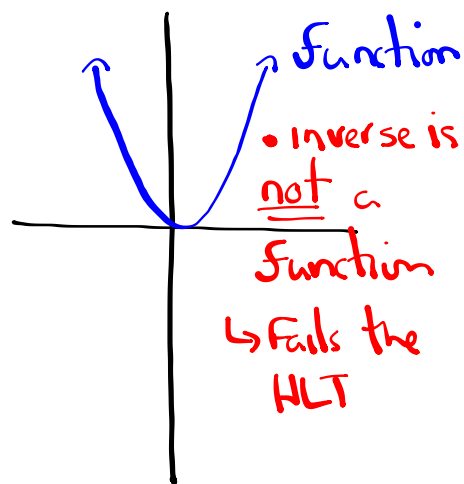
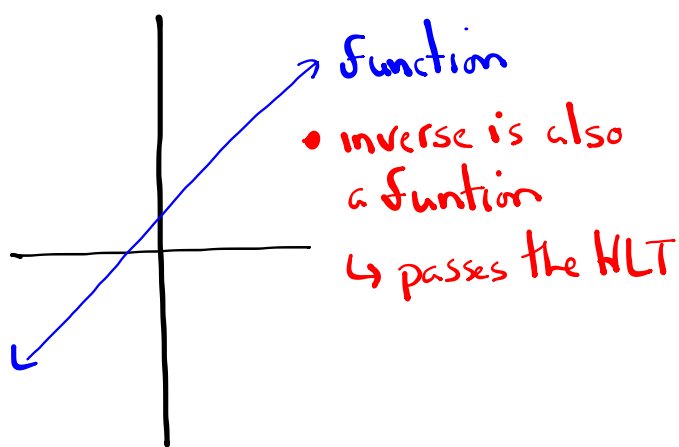


**horizontal line test**

- a test used to determine if the graph of an inverse relation will be a function
- if it is possible for a horizontal line to intersect the graph of a relation more than once, then the inverse of the relation is not a function

This is a function because it passes the vertical line test (VLT)

The inverse relation is not a function of x because it fails the vertical line test. There is more than one value of y in the range for at least one value of x in the domain. You can confirm this by using the **horizontal line test** on the graph of the original relation.

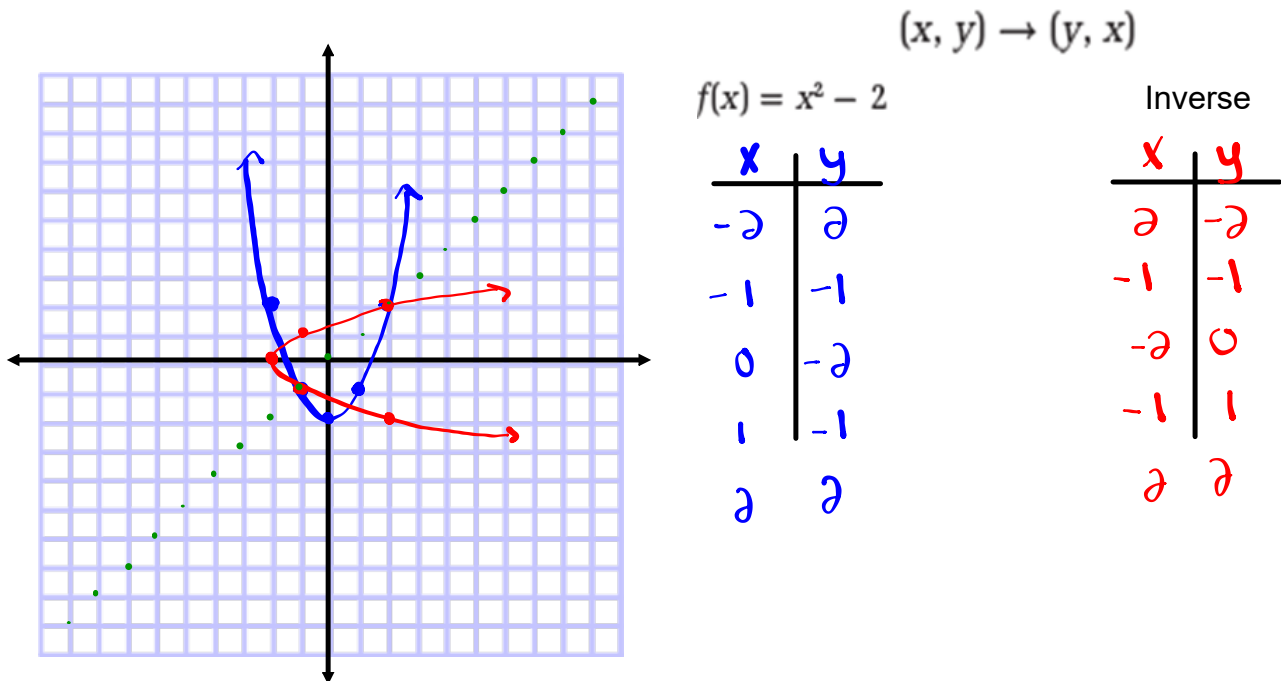


## Example 2

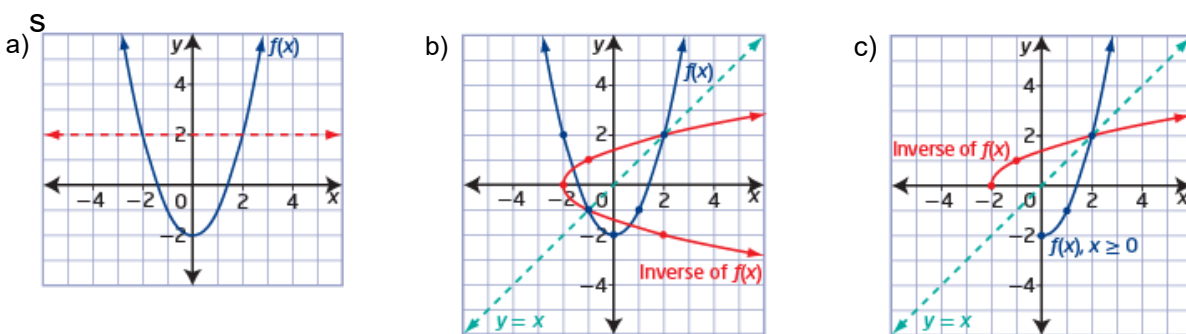
### Restrict the Domain

Consider the function  $f(x) = x^2 - 2$ .

- Graph the function  $f(x)$ . Is the inverse of  $f(x)$  a function? **No (Fails HLT)**
- Graph the inverse of  $f(x)$  on the same set of coordinate axes.
- Describe how the domain of  $f(x)$  could be restricted so that the inverse of  $f(x)$  is a function.



### Solution



- c) The inverse of  $f(x)$  is a function if the graph of  $f(x)$  passes the horizontal line test.

One possibility is to restrict the domain of  $f(x)$  so that the resulting graph is only one half of the parabola. Since the equation of the axis of symmetry is  $x = 0$ , restrict the domain to  $\{x \mid x \geq 0, x \in \mathbb{R}\}$ .

### Example 3

#### Determine the Equation of the Inverse

Algebraically determine the equation of the inverse of each function.

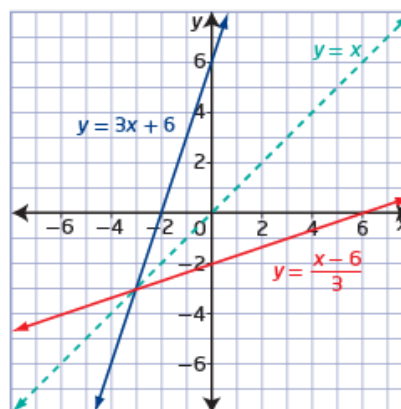
Verify graphically that the relations are inverses of each other.

a)  $f(x) = 3x + 6$

b)  $f(x) = x^2 - 4$

- 1) Replace  $f(x)$  with  $y$ .
- 2) Switch  $x$ 's and  $y$ 's.
- 3) Solve for  $y$ .
- 4) Replace  $y$  with  $f^{-1}(x)$ .  
(if the inverse is a function!)

Graph  $y = 3x + 6$  and  $y = \frac{x - 6}{3}$  on the same set of coordinate axes.



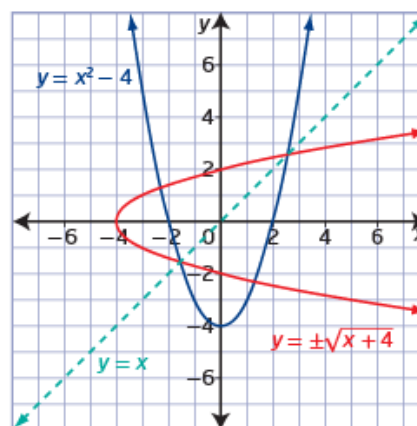
## Determine the Equation of the Inverse

b)  $f(x) = x^2 - 4$

- 1) Replace  $f(x)$  with  $y$ .
- 2) Switch  $x$ 's and  $y$ 's.
- 3) Solve for  $y$ .
- 4) Replace  $y$  with  $f^{-1}(x)$ .  
**(if the inverse is a function!)**

Why is this  $y$  not replaced with  $f^{-1}(x)$ ? What could be done so that  $f^{-1}(x)$  could be used?

Graph  $y = x^2 - 4$  and  $y = \pm\sqrt{x+4}$  on the same set of coordinate axes.

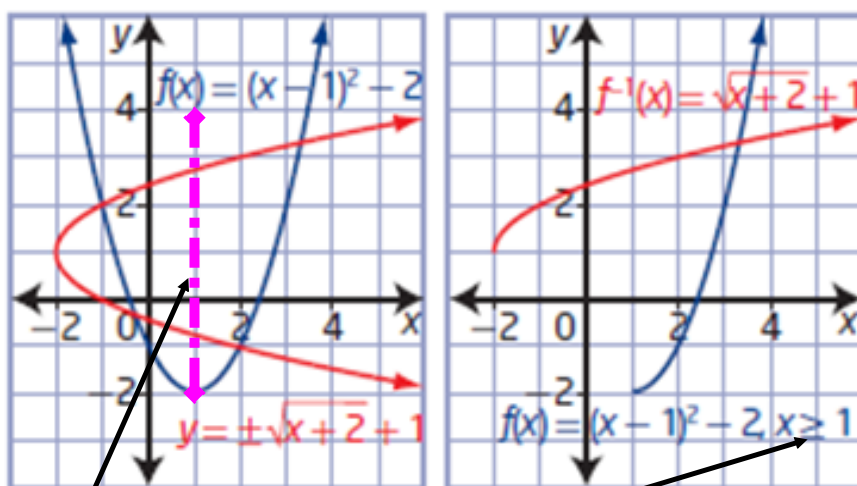




## Another example of how to restrict the domain

f)  $y = \pm\sqrt{x+2} + 1$

restricted domain  
 $\{x \mid x \geq 1, x \in \mathbb{R}\}$



axis of symmetry;  
 $x=1$

## Inverse of a Relation

### Key Ideas

- You can find the inverse of a relation by interchanging the  $x$ -coordinates and  $y$ -coordinates of the graph.
- The graph of the inverse of a relation is the graph of the relation reflected in the line  $y = x$ .
- The domain and range of a relation become the range and domain, respectively, of the inverse of the relation.
- Use the horizontal line test to determine if an inverse will be a function.
- You can create an inverse that is a function over a specified interval by restricting the domain of a function.
- When the inverse of a function  $f(x)$  is itself a function, it is denoted by  $f^{-1}(x)$ .
- You can verify graphically whether two functions are inverses of each other.

## Homework

Practice Problems...

Pages 51 - 55

#2, 3, 5, 6, 8, 9, 11, 15, 18, 20,  
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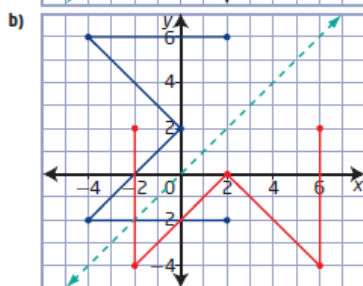
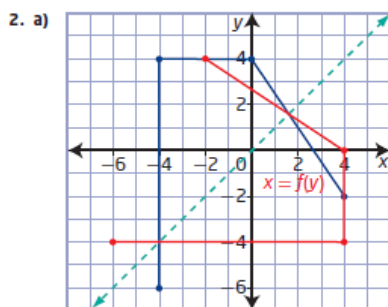
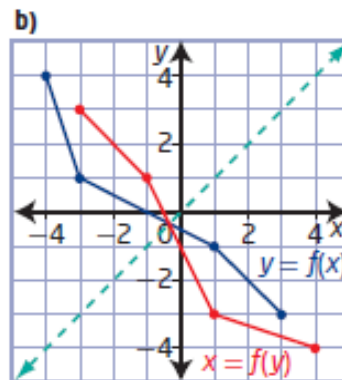
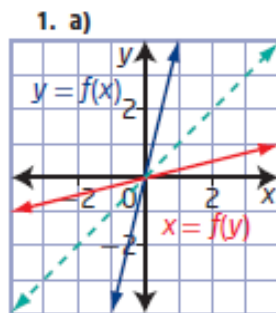
What if given the function algebraically?

Determine algebraically the equation of the inverse of each function.

**a)**  $f(x) = 3x - 6$       **b)**  $f(x) = \frac{1}{2}x + 5$

**c)**  $f(x) = \frac{1}{3}(x + 12)$       **d)**  $f(x) = \frac{8x + 12}{4}$

1.4 Inverse of a Relation, pages 51 to 55

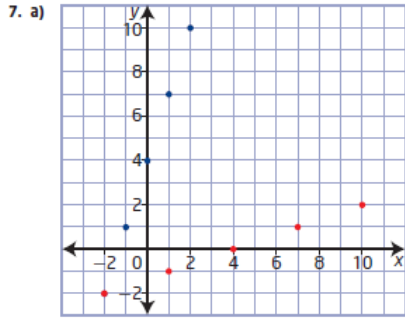


3. a) The graph is a function but the inverse will be a relation.  
 b) The graph and its inverse are functions.  
 c) The graph and its inverse are relations.

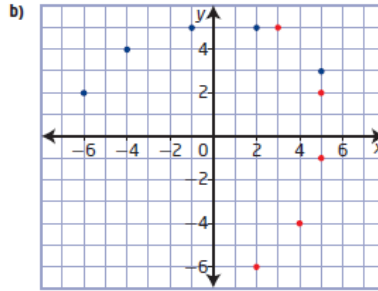
4. Examples:

- a)  $\{x \mid x \geq 0, x \in \mathbb{R}\}$  or  $\{x \mid x \leq 0, x \in \mathbb{R}\}$   
 b)  $\{x \mid x \geq -2, x \in \mathbb{R}\}$  or  $\{x \mid x \leq -2, x \in \mathbb{R}\}$   
 c)  $\{x \mid x \geq 4, x \in \mathbb{R}\}$  or  $\{x \mid x \leq 4, x \in \mathbb{R}\}$   
 d)  $\{x \mid x \geq -4, x \in \mathbb{R}\}$  or  $\{x \mid x \leq -4, x \in \mathbb{R}\}$   
 5. a)  $f^{-1}(x) = \frac{1}{7}x$       b)  $f^{-1}(x) = -\frac{1}{3}(x - 4)$   
 c)  $f^{-1}(x) = 3x - 4$       d)  $f^{-1}(x) = 3x + 15$   
 e)  $f^{-1}(x) = -\frac{1}{2}(x - 5)$       f)  $f^{-1}(x) = 2x - 6$

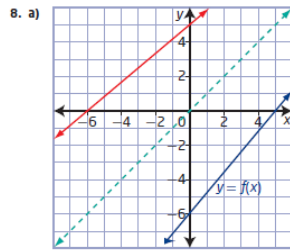
6. a) E    b) C    c) B    d) A    e) D



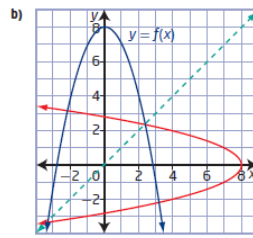
function: domain  $\{-2, -1, 0, 1, 2\}$ ,  
range  $\{-2, 1, 4, 7, 10\}$   
inverse: domain  $\{-2, 1, 4, 7, 10\}$ ,  
range  $\{-2, -1, 0, 1, 2\}$



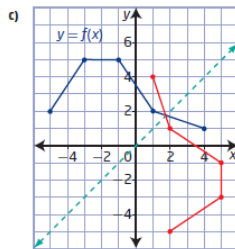
function: domain  $\{-6, -4, -1, 2, 5\}$ , range  $\{2, 3, 4, 5\}$   
inverse: domain  $\{2, 3, 4, 5\}$ , range  $\{-6, -4, -1, 2, 5\}$



The inverse is a function; it passes the vertical line test.

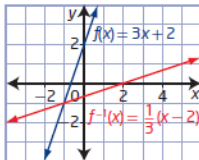


The inverse is not a function; it does not pass the vertical line test.



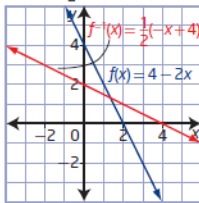
The inverse is not a function; it does not pass the vertical line test.

9. a)  $f^{-1}(x) = \frac{1}{3}(x - 2)$



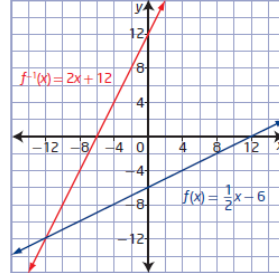
$f(x)$ : domain  $\{x \mid x \in \mathbb{R}\}$ ,  
range  $\{y \mid y \in \mathbb{R}\}$   
 $f^{-1}(x)$ : domain  $\{x \mid x \in \mathbb{R}\}$ ,  
range  $\{y \mid y \in \mathbb{R}\}$

b)  $f^{-1}(x) = \frac{1}{2}(-x + 4)$



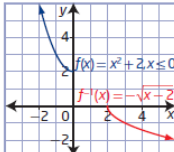
$f(x)$ : domain  $\{x \mid x \in \mathbb{R}\}$ ,  
range  $\{y \mid y \in \mathbb{R}\}$   
 $f^{-1}(x)$ : domain  $\{x \mid x \in \mathbb{R}\}$ ,  
range  $\{y \mid y \in \mathbb{R}\}$

c)  $f^{-1}(x) = 2x + 12$



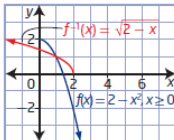
$f(x)$ : domain  $\{x \mid x \in \mathbb{R}\}$ , range  $\{y \mid y \in \mathbb{R}\}$   
 $f^{-1}(x)$ : domain  $\{x \mid x \in \mathbb{R}\}$ , range  $\{y \mid y \in \mathbb{R}\}$

d)  $f^{-1}(x) = -\sqrt{x-2}$



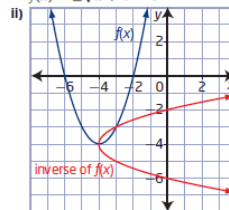
$f(x)$ : domain  $\{x \mid x \leq 0, x \in \mathbb{R}\}$ ,  
range  $\{y \mid y \geq 2, y \in \mathbb{R}\}$   
 $f^{-1}(x)$ : domain  $\{x \mid x \geq 2, x \in \mathbb{R}\}$ ,  
range  $\{y \mid y \leq 0, y \in \mathbb{R}\}$

e)  $f^{-1}(x) = \sqrt{2-x}$

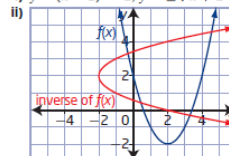


$f(x)$ : domain  $\{x \mid x \geq 0, x \in \mathbb{R}\}$ ,  
range  $\{y \mid y \leq 2, y \in \mathbb{R}\}$   
 $f^{-1}(x)$ : domain  $\{x \mid x \leq 2, x \in \mathbb{R}\}$ ,  
range  $\{y \mid y \geq 0, y \in \mathbb{R}\}$

10. a) i)  $f(x) = (x + 4)^2 - 4$ , inverse of  $f(x) = \pm\sqrt{x+4} - 4$

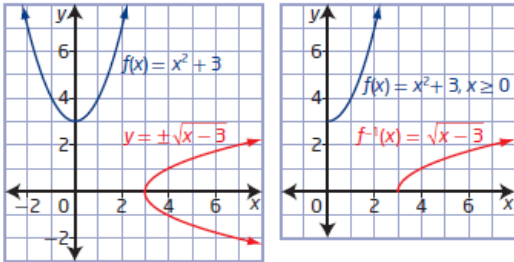


ii)  $y = (x - 2)^2 - 2, y = \pm\sqrt{x+2} + 2$

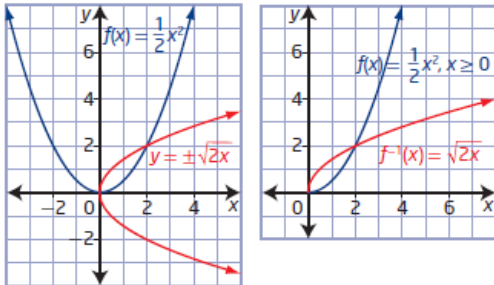


11. Yes, the graphs are reflections of each other in the line  $y = x$ .

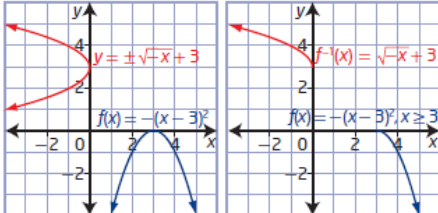
12. a)  $y = \pm\sqrt{x-3}$  restricted domain  $\{x \mid x \geq 0, x \in \mathbb{R}\}$



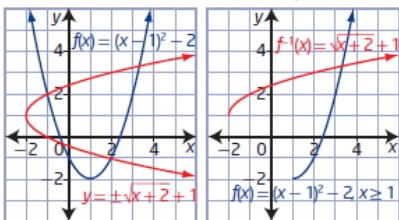
b)  $y = \pm\sqrt{2x}$  restricted domain  $\{x \mid x \geq 0, x \in \mathbb{R}\}$



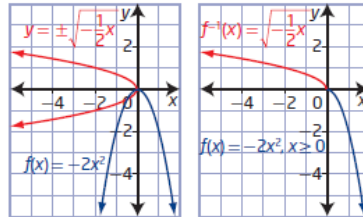
e)  $y = \pm\sqrt{-x+3}$  restricted domain  $\{x \mid x \geq 3, x \in \mathbb{R}\}$



f)  $y = \pm\sqrt{x+2} + 1$  restricted domain  $\{x \mid x \geq 1, x \in \mathbb{R}\}$



c)  $y = \pm\sqrt{-\frac{1}{2}x}$  restricted domain  $\{x \mid x \geq 0, x \in \mathbb{R}\}$



d)  $y = \pm\sqrt{x-1}$  restricted domain  $\{x \mid x \geq -1, x \in \mathbb{R}\}$

