Inverse of a Relation

An inverse function is a second function which undoes the work of the first one.

1. Introduction

Suppose we have a function f that takes x to y, so that

$$f(x) = y.$$

An inverse function, which we call f^{-1}, is another function that takes y back to x. So

$$f^{-1}(y) = x.$$

For f^{-1} to be an inverse of f, this needs to work for every x that f acts upon.

Inverse of a Relation

The inverse of a relation is found by interchanging the x-coordinates and y-coordinates of the ordered pairs of the relation. In other words, for every ordered pair (x, y) of a relation, there is an ordered pair (y, x) on the inverse of the relation. This means that the graphs of a relation and its inverse are reflections of each other in the line $y = x$.

$(x, y) \rightarrow (y, x)$ In plain English....the x and y coordinates will just switch places

$(-4, 2) \rightarrow (2, -4)$

The inverse of a function $y = f(x)$ may be written in the form $x = f(y)$. The inverse of a function is not necessarily a function. When the inverse of f is itself a function, it is denoted as f^{-1} and read as "f inverse." When the inverse of a function is not a function, it may be possible to restrict the domain to obtain an inverse function for a portion of the original function.

The inverse of a function reverses the processes represented by that function. Functions $f(x)$ and $g(x)$ are inverses of each other if the operations of $f(x)$ reverse all the operations of $g(x)$ in the opposite order and the operations of $g(x)$ reverse all the operations of $f(x)$ in the opposite order.

For example, $f(x) = 2x + 1$ multiplies the input value by 2 and then adds 1. The inverse function subtracts 1 from the input value and then divides by 2. The inverse function is $f^{-1}(x) = \frac{x-1}{2}$.

Function	Inverse function
$f(x) = 2x+1$	$f^{-1}(x) = \frac{x-1}{2}$
$f(1) = 2(1)+1$	$f^{-1}(3) = \frac{3-1}{2} = \frac{2}{2} = 1$
$f(1) = 2+1$	
$f(1) = 3$	
$\boxed{(1,3)} \longrightarrow$	$\boxed{(3,1)}$

Example 1

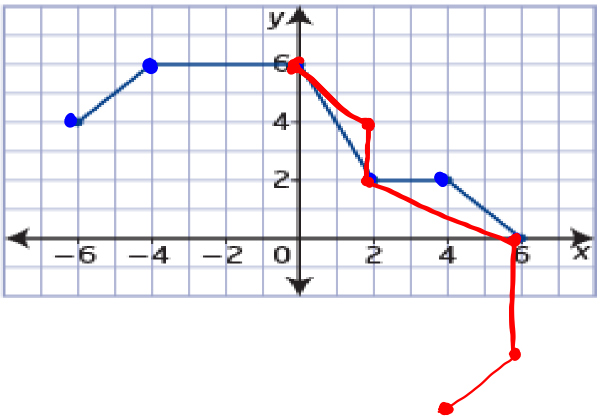
Graph an Inverse

Consider the graph of the relation shown.

a) Sketch the graph of the inverse relation.

b) State the domain and range of the relation and its inverse.

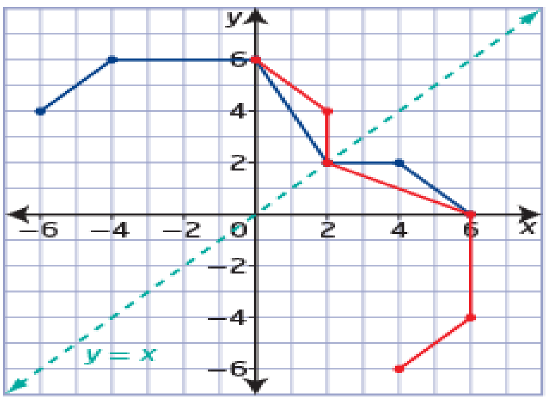
c) Determine whether the relation and its inverse are functions.



Solution

a) To graph the inverse relation, interchange the x-coordinates and y-coordinates of key points on the graph of the relation.

Points on the Relation	Points on the Inverse Relation
(−6, 4)	(4, −6)
(−4, 6)	(6, −4)
(0, 6)	(6, 0)
(2, 2)	(2, 2) * Invariant
(4, 2)	(2, 4)
(6, 0)	(0, 6)



The graphs are reflections of each other in the line $y = x$. The points on the graph of the relation are related to the points on the graph of the inverse relation by the mapping $(x, y) \to (y, x)$.

What points are invariant after a reflection in the line $y = x$?

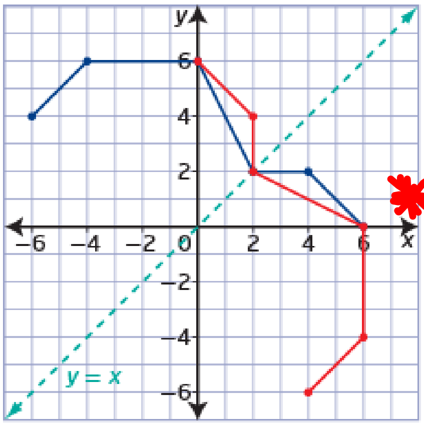
↳ any point where $y=x$

ex: (2,2)

(5,5)

(-10,-10)

b) State the domain and range of the relation and its inverse.



[-6,6] [0,6]

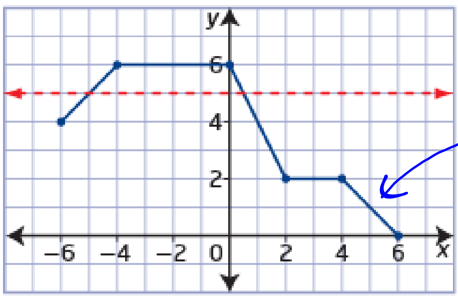
	Domain	Range
Relation	$\{x \mid -6 \le x \le 6, x \in R\}$	$\{y \mid 0 \le y \le 6, y \in R\}$
Inverse Relation	$\{x \mid 0 \le x \le 6, x \in R\}$	$\{y \mid -6 \le y \le 6, y \in R\}$

[0,6] [-6,6]

The domain of the relation becomes the range of the inverse relation and the range of the relation becomes the domain of the inverse relation.

In plain English....the x and y coordinates will just switch places

c) Determine whether the relation and its inverse are functions.

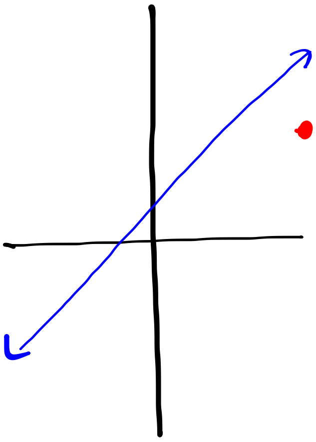


This is a function because it passes the vertical line test (VLT)

horizontal line test

- a test used to determine if the graph of an inverse relation will be a function
- if it is possible for a horizontal line to intersect the graph of a relation more than once, then the inverse of the relation is not a function

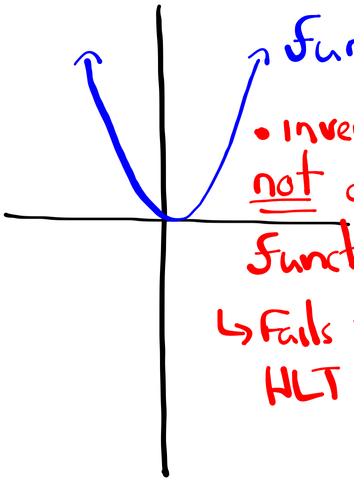
The inverse relation is not a function of x because it fails the vertical line test. There is more than one value of y in the range for at least one value of x in the domain. You can confirm this by using the **horizontal line test** on the graph of the original relation.



Function

- inverse is also a funtion

↳ passes the HLT



Function

- inverse is <u>not</u> a function

↳ Fails the HLT

Example 2

Restrict the Domain

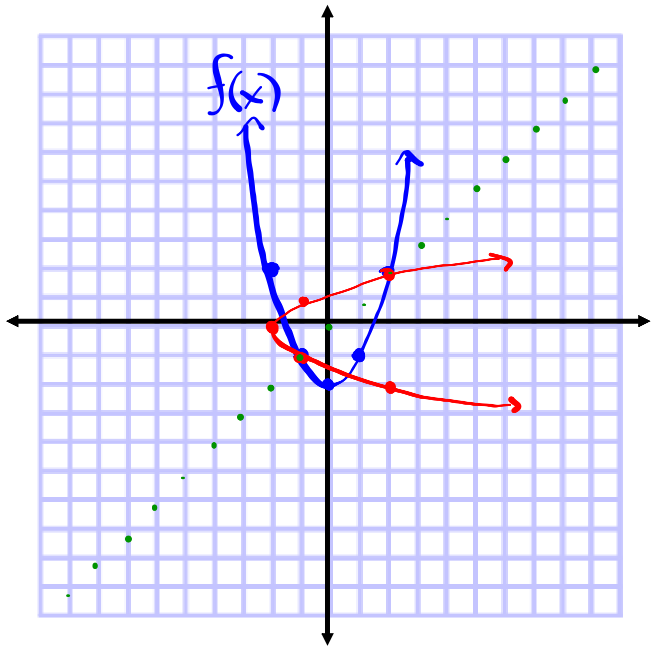
Consider the function $f(x) = x^2 - 2$.

a) Graph the function $f(x)$. Is the inverse of $f(x)$ a function? No (Fails HLT)

b) Graph the inverse of $f(x)$ on the same set of coordinate axes.

c) Describe how the domain of $f(x)$ could be restricted so that the inverse of $f(x)$ is a function.

$(x, y) \rightarrow (y, x)$



$f(x) = x^2 - 2$

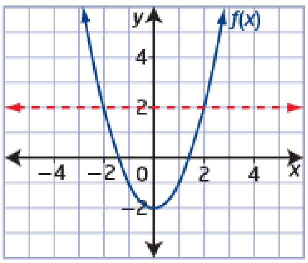
x	y
-2	2
-1	-1
0	-2
1	-1
2	2

Inverse

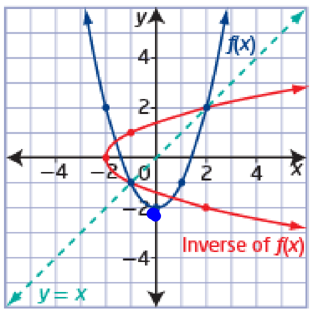
x	y
2	-2
-1	-1
-2	0
-1	1
2	2

Solutions

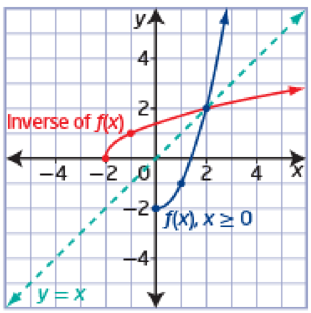
a)



b)



c)



Vertex: $(0, -2)$

axis of symmetry: $x = 0$

c) The inverse of $f(x)$ is a function if the graph of $f(x)$ passes the horizontal line test.

One possibility is to restrict the domain of $f(x)$ so that the resulting graph is only one half of the parabola. Since the equation of the axis of symmetry is $x = 0$, restrict the domain to $\{x \mid x \geq 0, x \in R\}$. $[0, \infty)$

restrict domain to $\rightarrow$

or $\{x \mid x \leq 0, x \in R\}$ $(-\infty, 0]$

Example 3

Determine the Equation of the Inverse

Algebraically determine the equation of the inverse of each function. Verify graphically that the relations are inverses of each other.

a) $f(x) = 3x + 6$

b) $f(x) = x^2 - 4$

a) $f(x) = 3x + 6$ (Linear)

$y = 3x + 6$

$x = 3y + 6$

$\frac{x - 6}{3} = \frac{3y}{3}$

$\frac{x-6}{3} = y$

$y = \frac{x-6}{3}$

$\boxed{f^{-1}(x) = \frac{x-6}{3}}$

or $y = \frac{x}{3} - \frac{6}{3}$

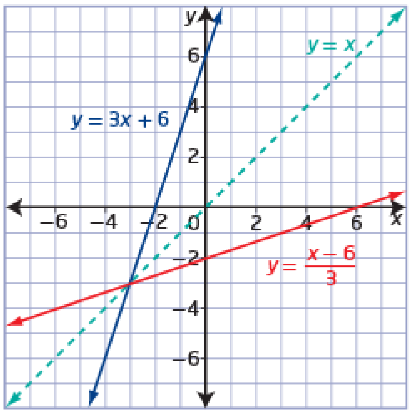
$y = \frac{x}{3} - 2$

or

$\boxed{f^{-1}(x) = \frac{x}{3} - 2}$

1) Replace $f(x)$ with y.
2) Switch x's and y's.
3) Solve for y.
4) Replace y with $f^{-1}(x)$.
(if the inverse is a function!)

Graph $y = 3x + 6$ and $y = \frac{x-6}{3}$ on the same set of coordinate axes.



$\frac{x}{3} = \frac{3y + 6}{3}$

$\frac{x}{3} = y + 2$

$\frac{x}{3} - 2 = y$

Determine the Equation of the Inverse

b) $f(x) = x^2 - 4$ (Quadratic) ∪

1) Replace $f(x)$ with y.
2) Switch x's and y's.
3) Solve for y.
4) Replace y with $f^{-1}(x)$. (if the inverse is a function!)

$y = x^2 - 4$

$x = y^2 - 4$

$x + 4 = y^2$

$\pm\sqrt{x+4} = y$

$y = \pm\sqrt{x+4}$

Why is this y not replaced with $f^{-1}(x)$? What could be done so that $f^{-1}(x)$ could be used?

Graph $y = x^2 - 4$ and $y = \pm\sqrt{x+4}$ on the same set of coordinate axes.

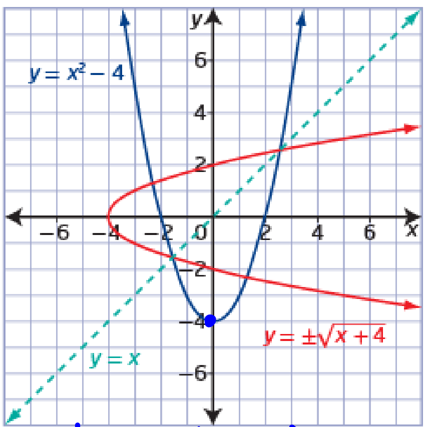
you could restrict domain on $f(x)$

- if $f(x) = x^2 - 4,\ x \geq 0$

  $f^{-1}(x) = \sqrt{x+4}$

- if $f(x) = x^2 - 4,\ x \leq 0$

  $f^{-1}(x) = -\sqrt{x+4}$



Vertex: $(\underline{0}, -4)$

Axis of Sym: $x = 0$

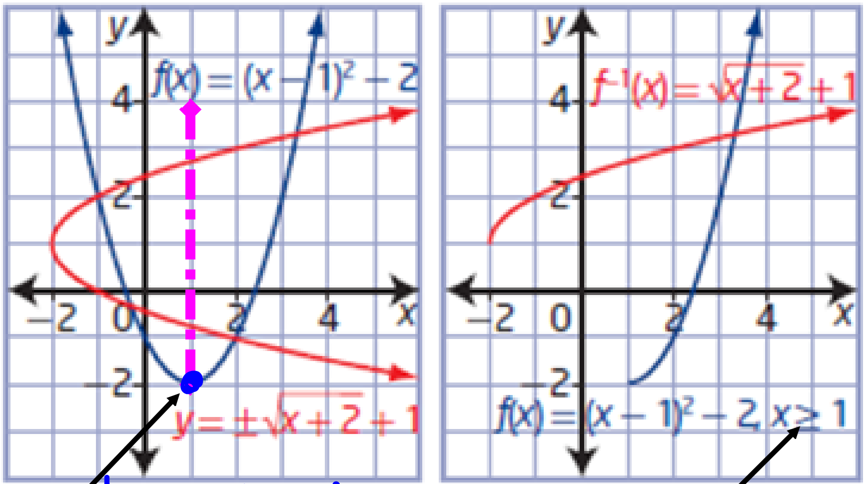
Restriction: $x \geq 0$

or $x \leq 0$

Another example of how to restrict the domain

f) $y = \pm\sqrt{x+2} + 1$

restricted domain
$\{x \mid x \geq 1, x \in R\}$



Vertex: $(\underline{1}, -2)$

axis of sym: $x = 1$

Restriction: $x \geq 1$

Inverse of a Relation

Key Ideas

- You can find the inverse of a relation by interchanging the x-coordinates and y-coordinates of the graph.
- The graph of the inverse of a relation is the graph of the relation reflected in the line $y = x$.
- The domain and range of a relation become the range and domain, respectively, of the inverse of the relation.
- Use the horizontal line test to determine if an inverse will be a function.
- You can create an inverse that is a function over a specified interval by restricting the domain of a function.
- When the inverse of a function $f(x)$ is itself a function, it is denoted by $f^{-1}(x)$.
- You can verify graphically whether two functions are inverses of each other.

Homework

Practice Problems...

Pages 51 - 55
#2, 3, 5, 6, 8, 9, 11, 15, 18, 20, 21

15. Given the function $f(x) = 4x - 2$, (Linear ↗)
determine each of the following.

a) $f^{-1}(4)$
b) $f^{-1}(-2)$
c) $f^{-1}(8)$
d) $f^{-1}(0)$

(1) $f(x) = 4x - 2$

$y = 4x - 2$

$x = 4y - 2$

$\frac{x+2}{4} = \frac{4y}{4}$

$\frac{x+2}{4} = y$

$y = \frac{x+2}{4}$

$\boxed{f^{-1}(x) = \frac{x+2}{4}}$

$f^{-1}(x) = \frac{1}{4}(x+2)$

$f^{-1}(x) = \frac{x}{4} + \frac{1}{2}$

a) $f^{-1}(4) = \frac{(4)+2}{4} = \frac{6}{4} = \frac{3}{2}$ $(4, 3/2)$

b) $f^{-1}(-2) = \frac{(-2)+2}{4} = \frac{0}{4} = 0$ $(-2, 0)$

18. In Canada, ring sizes are specified using a numerical scale. The numerical ring size, y, is approximately related to finger circumference, x, in millimetres, by $y = \frac{x - 36.5}{2.55}$.

a) What whole-number ring size corresponds to a finger circumference of 49.3 mm? $(x = 49.3)$

b) Determine an equation for the inverse of the function. What do the variables represent?

c) What finger circumferences correspond to ring sizes of 6, 7, and 9?

a) $y = \frac{x - 36.5}{2.55}$

$y = \frac{49.3 - 36.5}{2.55}$

$y = \frac{12.8}{2.55} = 5.019$

$\boxed{y = 5}$

b) $y = \frac{x - 36.5}{2.55}$

$2.55x = \frac{y - 36.5}{\cancel{2.55}} \cdot \cancel{2.55}$

$2.55x = y - 36.5$

$2.55x + 36.5 = y$

$\boxed{y = 2.55x + 36.5}$

$f^{-1}(x) = 2.55x + 36.5$

c) $f^{-1}(6) = 2.55(6) + 36.5$

$= 15.3 + 36.5$

$= 51.8$ mm

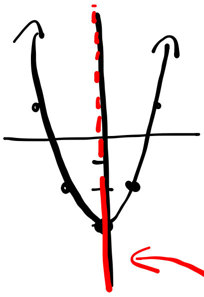
VSF = 5 → $a = 5$ reflection in x-axis ($a<0$)

HSF = $\frac{1}{2}$ → $b = 2$ " in y-axis ($b<0$)

translated 3 left → $h = -3$

" 4 up → $k = 4$

$f(x) = x^2 - 3$



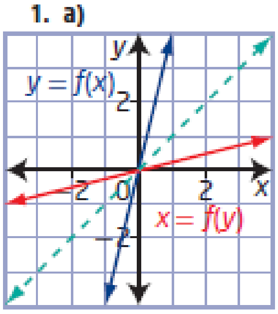
axis of symmetry $x=0$

restrict domain of $f(x)$

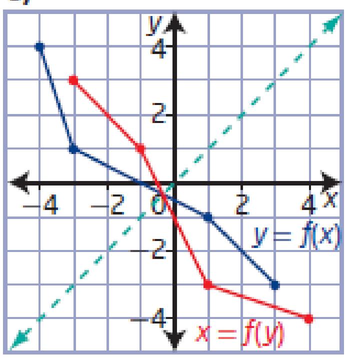
to $\{x \mid x \geq 0, x \in R\}$

1.4 Inverse of a Relation, pages 51 to 55

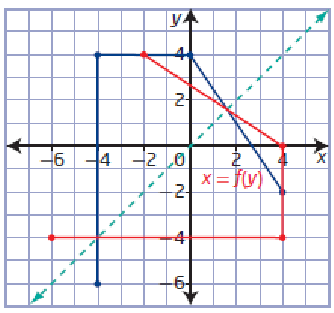
1. a)



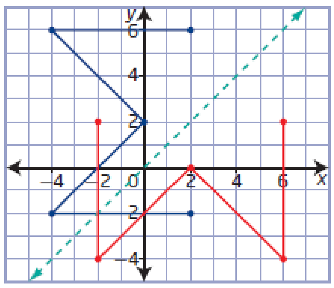
b)



2. a)



b)



3. a) The graph is a function but the inverse will be a relation.

b) The graph and its inverse are functions.

c) The graph and its inverse are relations.

4. Examples:

a) $\{x \mid x \geq 0, x \in \mathbb{R}\}$ or $\{x \mid x \leq 0, x \in \mathbb{R}\}$

b) $\{x \mid x \geq -2, x \in \mathbb{R}\}$ or $\{x \mid x \leq -2, x \in \mathbb{R}\}$

c) $\{x \mid x \geq 4, x \in \mathbb{R}\}$ or $\{x \mid x \leq 4, x \in \mathbb{R}\}$

d) $\{x \mid x \geq -4, x \in \mathbb{R}\}$ or $\{x \mid x \leq -4, x \in \mathbb{R}\}$

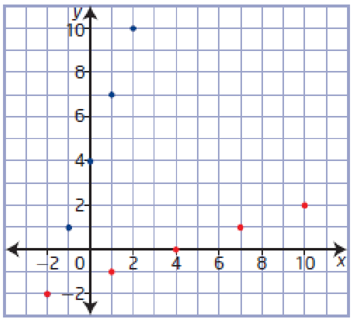
5. a) $f^{-1}(x) = \frac{1}{7}x$ **b)** $f^{-1}(x) = -\frac{1}{3}(x - 4)$

c) $f^{-1}(x) = 3x - 4$ **d)** $f^{-1}(x) = 3x + 15$

e) $f^{-1}(x) = -\frac{1}{2}(x - 5)$ **f)** $f^{-1}(x) = 2x - 6$

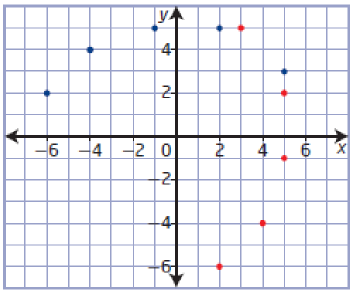
6. a) E **b)** C **c)** B **d)** A **e)** D

7. a)



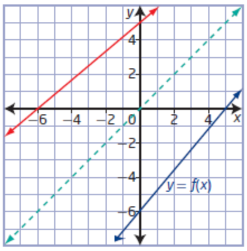
function: domain $\{-2, -1, 0, 1, 2\}$, range $\{-2, 1, 4, 7, 10\}$
inverse: domain $\{-2, 1, 4, 7, 10\}$, range $\{-2, -1, 0, 1, 2\}$

b)



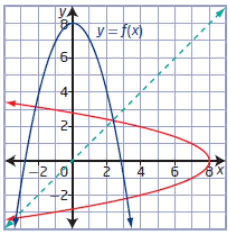
function: domain $\{-6, -4, -1, 2, 5\}$, range $\{2, 3, 4, 5\}$
inverse: domain $\{2, 3, 4, 5\}$, range $\{-6, -4, -1, 2, 5\}$

8. a)



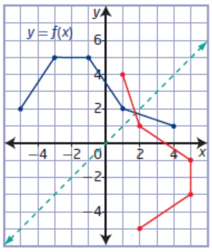
The inverse is a function; it passes the vertical line test.

b)



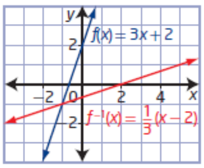
The inverse is not a function; it does not pass the vertical line test.

c)



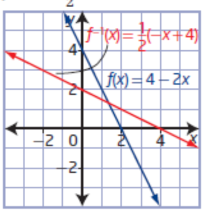
The inverse is not a function; it does not pass the vertical line test.

9. a) $f^{-1}(x) = \frac{1}{3}(x - 2)$



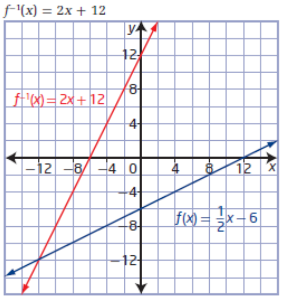
$f(x)$: domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$
$f^{-1}(x)$: domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$

b) $f^{-1}(x) = \frac{1}{2}(-x + 4)$



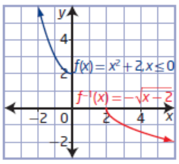
$f(x)$: domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$
$f^{-1}(x)$: domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$

c) $f^{-1}(x) = 2x + 12$



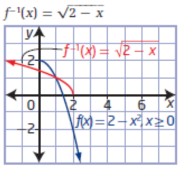
$f(x)$: domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$
$f^{-1}(x)$: domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$

d) $f^{-1}(x) = -\sqrt{x - 2}$



$f(x)$: domain $\{x \mid x \leq 0, x \in \mathrm{R}\}$, range $\{y \mid y \geq 2, y \in \mathrm{R}\}$
$f^{-1}(x)$: domain $\{x \mid x \geq 2, x \in \mathrm{R}\}$, range $\{y \mid y \leq 0, y \in \mathrm{R}\}$

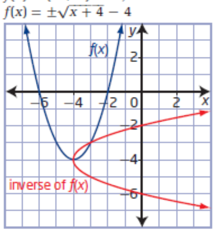
e) $f^{-1}(x) = \sqrt{2 - x}$



$f(x)$: domain $\{x \mid x \geq 0, x \in \mathrm{R}\}$, range $\{y \mid y \leq 2, y \in \mathrm{R}\}$
$f^{-1}(x)$: domain $\{x \mid x \leq 2, x \in \mathrm{R}\}$, range $\{y \mid y \geq 0, y \in \mathrm{R}\}$

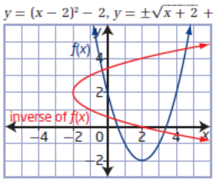
10. a) i) $f(x) = (x + 4)^2 - 4$, inverse of $f(x) = \pm\sqrt{x + 4} - 4$

ii)



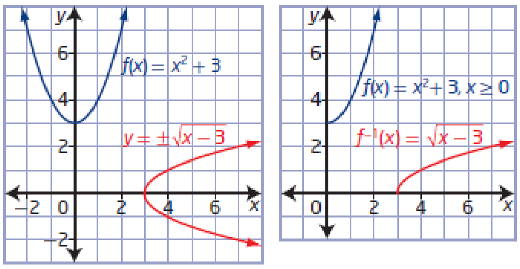
b) i) $y = (x - 2)^2 - 2$, $y = \pm\sqrt{x + 2} + 2$

ii)

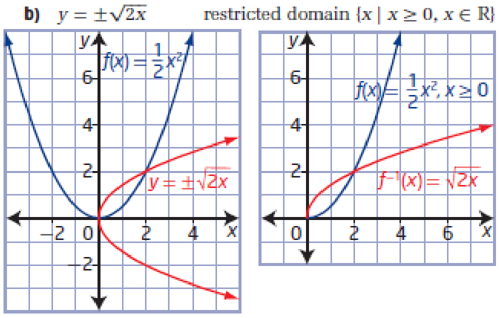


11. Yes, the graphs are reflections of each other in the line $y = x$.

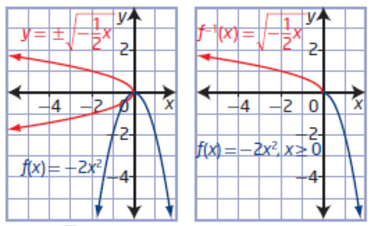
12. a) $y = \pm\sqrt{x - 3}$ restricted domain $\{x \mid x \geq 0, x \in \mathrm{R}\}$



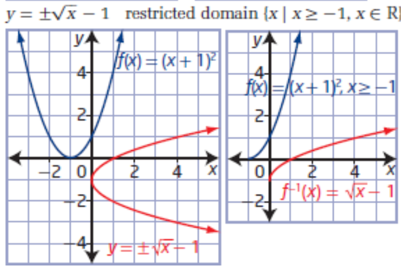
b) $y = \pm\sqrt{2x}$ restricted domain $\{x \mid x \geq 0, x \in \mathrm{R}\}$



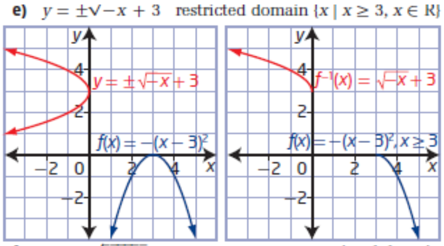
c) $y = \pm\sqrt{-\frac{1}{2}x}$ restricted domain $\{x \mid x \geq 0, x \in \mathrm{R}\}$



d) $y = \pm\sqrt{x} - 1$ restricted domain $\{x \mid x \geq -1, x \in \mathrm{R}\}$



e) $y = \pm\sqrt{-x} + 3$ restricted domain $\{x \mid x \geq 3, x \in \mathrm{R}\}$



f) $y = \pm\sqrt{x + 2} + 1$ restricted domain $\{x \mid x \geq 1, x \in \mathrm{R}\}$