Questions from Homework

8) $f(3) = 33$ $\boxed{f(x) = 4x^2 - 2x + 3}$

$f'(3) = 22$ $f'(x) = 8x - 2$

$f''(3) = 8$ $f''(x) = 8$

4) $f(x) = \sqrt{1+x^3} = (1+x^3)^{1/2}$

$$f'(x) = \frac{1}{2}(1+x^3)^{-1/2}(3x^2) = \frac{3x^2}{2(1+x^3)^{1/2}}$$

$$f''(x) = \frac{6x(2)(1+x^3)^{1/2} - 3x^2(1+x^3)^{-1/2}(3x^2)}{\left[2(1+x^3)^{1/2}\right]^2}$$

$$f''(x) = \frac{12x(1+x^3)^{1/2} - 9x^4(1+x^3)^{-1/2}}{4(1+x^3)}$$

$4 + 4x^3 - 3x^3$

$$f''(x) = \frac{3x(1+x^3)^{-1/2}\left[4(1+x^3) - 3x^3\right]}{4(1+x^3)}$$

$$f''(x) = \frac{3x(4+x^3)}{4(1+x^3)^{3/2}} = \frac{3x(4+x^3)}{4\sqrt{(1+x^3)^3}}$$

$$f''(2) = \frac{6(\cancel{12}^{\,3})}{{}_1\cancel{4}\sqrt{729}} = \frac{\cancel{18}^{\,2}}{\cancel{27}_{\,3}} = \frac{2}{3}$$

Implicit Differentiation

So far we have described functions by expressing one variable *explicitly* in terms of another variable: for example,

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

- Sometimes an equation only <u>implicitly</u> defines y as a function (or functions) of x.
- Examples
  - $x^2 + y^2 = 25$
  - $x^3 + y^3 = 6xy$

$$x^2 + y^2 = 25$$
$$y^2 = 25 - x^2$$
$$y = \pm\sqrt{25 - x^2}$$

$$\cancel{(5-x)(5-x)}$$
$$\cancel{25 - 5x - 5x + x^2}$$
$$\cancel{25 - 10x + x^2}$$

- The first equation could easily be rearranged for y =...

$$y = \pm\sqrt{25 - x^2}$$

Actually gives two functions

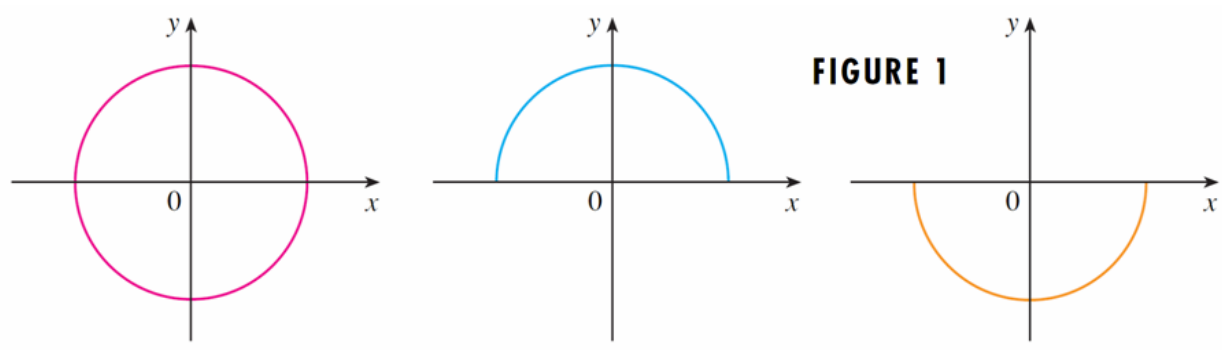

FIGURE 1

(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

Implicit Differentiation

- There is a way called *implicit differentiation* to find $\boxed{dy/dx}$ without solving for y :
  1. First differentiate both sides of the equation with respect to x ;
  2. Then solve the resulting equation for y' . $\left(\frac{dy}{dx}\right)$
- We will always assume that the given equation does indeed define y as a differentiable function of x .

Example

- For the circle $x^2 + y^2 = 25$, find
  - a) dy/dx
  - b) an equation of the tangent at the point $(3, 4)$.

$x_1 = 3 \quad y_1 = 4$

Solution:

Start by differentiating both sides of the equation:

a) $x^2 + y^2 = 25$

$$2(x)(1) + 2(y)\left(\frac{dy}{dx}\right) = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y} = -\frac{x}{y}$$

b) Find the equation of the tangent @ (3,4)

(i) Find m:

$$\frac{dy}{dx} = -\frac{x}{y} = \frac{-(3)}{(4)} = -\frac{3}{4} \leftarrow m$$

(ii) Find equation:

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -\frac{3}{4}(x - 3)$$

$$y - 4 = -\frac{3}{4}x + \frac{9}{4}$$

$$y = -\frac{3}{4}x + \frac{9}{4} + 4$$

$$\boxed{y = -\frac{3}{4}x + \frac{25}{4}} \quad \text{or} \quad 4y = -3x + 25$$

$$\boxed{3x + 4y - 25 = 0}$$

Same Example Revisited

- Since it is easy to solve this equation for y, we
  - do so, and then
  - find the equation of the tangent line by earlier methods, and then
  - compare the result with our preceding answer:

Solution

- Solving the equation gives $y = \pm\sqrt{25 - x^2}$ as before.
- The point (3, 4) lies on the <u>upper</u> semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$

Differentiate *f*:

(i) $y = (25 - x^2)^{1/2}$

$y' = \frac{1}{\cancel{2}}(25 - x^2)^{-1/2}(-\cancel{2}x) = \frac{-x}{\sqrt{25 - x^2}}$

(ii) $y' = \frac{-x}{\sqrt{25 - x^2}} = \frac{-(3)}{\sqrt{25 - (3)^2}} = \frac{-3}{4} \rightarrow m$

(iii) $y - y_1 = m(x - x_1)$

$y - 4 = -\frac{3}{4}(x - 3)$

$y - 4 = -\frac{3}{4}x + \frac{9}{4}$

$\boxed{y = -\frac{3}{4}x + \frac{25}{4}}$

Solution (cont'd)

- So $f'(3) = -\dfrac{3}{\sqrt{25-3^2}} = -\dfrac{3}{4}$,

  leading to the same equation

  $$3x + 4y = 25$$

  for the tangent that we obtained earlier.
- Note that although this problem <u>could</u> be done both ways, implicit differentiation was easier!

Sometimes Implicit Differentiation is not only the easiest way, it's the *only* way

Example:

Given $x^3 + y^3 = 6xy$

Product

$f(x)\ g(x)$

Find $\frac{dy}{dx}$

$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$

$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$

$\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$

$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$

$\frac{dy}{dx} = \frac{\cancel{3}(2y - x^2)}{\cancel{3}(y^2 - 2x)}$

$\frac{dy}{dx} = \boxed{\frac{2y - x^2}{y^2 - 2x}}$

Product

Given $$2x^5 + (x^4)(y) + y^5 = 36$$

Find $\frac{dy}{dx}$

$$10x^4 + 4x^3y + x^4\frac{dy}{dx} + 5y^4\frac{dy}{dx} = 0$$

$$x^4\frac{dy}{dx} + 5y^4\frac{dy}{dx} = -10x^4 - 4x^3y$$

$$\frac{dy}{dx}(x^4 + 5y^4) = -10x^4 - 4x^3y$$

$$\frac{dy}{dx} = \frac{-10x^4 - 4x^3y}{x^4 + 5y^4}$$

$$\frac{dy}{dx} = -\frac{10x^4 + 4x^3y}{x^5 + 5y^4}$$

$$\frac{dy}{dx} = -\frac{2x^3(5x + 2y)}{x^4 + 5y^4}$$

$$2(x)^5 + (y)^5$$

$$10(x)^4(1) + 5(y)^4\left(\frac{dy}{dx}\right)$$

$$10x^4 + 5y^4\frac{dy}{dx}$$

Homework

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Do # 1-5