Questions from Homework

Exercise 2.8

(4) If $f(x) = \sqrt{1+x^3}$ find $f''(2)$

$$f(x) = (1+x^3)^{1/2}$$

$$f'(x) = \frac{1}{2}(1+x^3)^{-1/2}(3x^2)$$

$$f'(x) = \frac{3x^2}{2(1+x^3)^{1/2}}$$

$$f''(x) = \frac{\overset{f'}{6x}\left(\overset{g}{2(1+x^3)^{1/2}}\right) - \overset{f}{3x^2}\,\overset{g'}{(1+x^3)^{-1/2}(3x^2)}}{4(1+x^3)}$$

$$f''(x) = \frac{12x(1+x^3)^{1/2} - 9x^4(1+x^3)^{-1/2}}{4(1+x^3)}$$

$4 + 4x^3 - 3x^3$

$$f''(x) = \frac{3x(1+x^3)^{-1/2}\left[4(1+x^3) - 3x^3\right]}{4(1+x^3)}$$

$$f''(x) = \frac{3x(4+x^3)}{4(1+x^3)^{3/2}}$$

$$f''(2) = \frac{3(2)(4+(2)^3)}{4(1+(2)^3)^{3/2}}$$

$$f''(2) = \frac{72}{108}$$

$$\boxed{f''(2) = \frac{2}{3}}$$

Questions from Homework

⑧ Find a quadratic function f such that

$f(3) = 33$

$f'(3) = 22$

$f''(3) = 8$

$\boxed{f(x) = 4x^2 - 2x + 3}$

$f'(x) = 8x - 2$

$f''(x) = 8$

Product Rule:

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$$

Quotient Rule:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Chain Rule:

$$(f \circ g)'(x) = f'(g(x))(g'(x))$$

Differentiation Rules

Product Rule:

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Express the product rule verbally if you are considering a function of the form...

$f(x)$= (First) x (Second)

" The derivative of the product of two functions is the the first multiplied by the derivative of second, plus the derivative of first multiplied by the second"

Get in the habit of verbalizing the rule as you differentiate...it will help when the functions get more complicated.

Quotient Rule:

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Express the quotient rule verbally ...

" The denominator multiplied by the derivative of the numerator, minus the numerator multiplied by the derivative of the denominator, all over the denominator squared"

Combining the Chain Rule With the Product and Quotient Rule:

The Chain Rule If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product

$$F'(x) = f'(g(x))g'(x)$$

Differentiate the following function and simplify your answer:

$$y = \underbrace{(x^2+1)^3}_{f(x)}\underbrace{(2-3x)^4}_{g(x)}$$

$$\frac{dy}{dx} = \underbrace{3(x^2+1)^2(2x)}_{f'(x)}\underbrace{(2-3x)^4}_{g(x)} + \underbrace{(x^2+1)^3}_{f(x)}\underbrace{(4)(2-3x)^3(-3)}_{g'(x)}$$

$$\frac{dy}{dx} = 6x(x^2+1)^2(2-3x)^4 - 12(x^2+1)^3(2-3x)^3$$

$2x \quad -3x^2 \qquad -2x^2 \quad -2$

$$\frac{dy}{dx} = 6(x^2+1)^2(2-3x)^3\left[x(2-3x) - 2(x^2+1)\right]$$

$$\frac{dy}{dx} = 6(x^2+1)^2(2-3x)^3(-5x^2+2x-2)$$

$$\boxed{\frac{dy}{dx} = -6(x^2+1)^2(2-3x)^3(5x^2-2x+2)}$$

Differentiate the following functions and simplify your answers:

$$s = \left(\frac{2t-1}{t+2}\right)^6$$

$2t+4 \quad -2t \quad +1$

$$\frac{ds}{dt} = 6\left(\frac{2t-1}{t+2}\right)^5 \left[\frac{2(t+2)-(2t-1)}{(t+2)^2}\right]$$

$$\frac{ds}{dt} = 6 \cdot \frac{(2t-1)^5}{(t+2)^5} \cdot \frac{5}{(t+2)^2}$$

$$\boxed{\frac{ds}{dt} = \frac{30(2t-1)^5}{(t+2)^7}}$$

Example 1

Chain Rule

Let $F(x) = f(g(x))$

If $f(2) = 3$, $f'(2) = 5$, $g(1) = 2$ and $g'(1) = 4$ find $F'(1)$.

$F'(x) = f'(g(x))g'(x)$

$F'(1) = f'(g(1))g'(1)$

$F'(1) = f'(2) \cdot (4)$

$F'(1) = (5) \cdot (4)$

$\boxed{F'(1) = 20}$

Example 2

If $y = u^{10} + u^5 + 2$, where $u = 1 - 3x^2$, find $\left.\frac{dy}{dx}\right]_{x=1}$

① $\frac{dy}{du} = 10u^9 + 5u^4$

② $\frac{du}{dx} = -6x$

③ when $x = 1$

$u = 1 - 3x^2$

$u = 1 - 3(1)^2$

$u = -2$

④ $\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$

$\left.\frac{dy}{dx}\right]_{x=1} = [10u^9 + 5u^4][-6x]$

$\left.\frac{dy}{dx}\right]_{x=1} = [10(-2)^9 + 5(-2)^4][-6(1)]$

$\left.\frac{dy}{dx}\right]_{x=1} = [-5120 + 80][-6]$

$\left.\frac{dy}{dx}\right]_{x=1} = [-5040][-6]$

$\left.\frac{dy}{dx}\right]_{x=1} = 30240$

Homework

Page 112

#4-9, 11, 12, and 14

Page 112

4 e) $y = \sqrt{x}(5 - \sqrt{x}) = 5\sqrt{x} - x = 5x^{1/2} - x$

$y' = \frac{5}{2}x^{-1/2} - 1$

$y' = \frac{5}{2\sqrt{x}} - 1 = \frac{5 - 2\sqrt{x}}{2\sqrt{x}}$

Using Product:

$y = \sqrt{x}(5 - \sqrt{x})$ $\qquad y' = \frac{1}{2}x^{-1/2}(5 - \sqrt{x}) + x^{1/2}\left(-\frac{1}{2}x^{-1/2}\right)$

$y' = \frac{1}{2\sqrt{x}}(5 - \sqrt{x}) + \sqrt{x}\left(-\frac{1}{2\sqrt{x}}\right)$

$y' = \frac{5}{2\sqrt{x}} - \frac{\sqrt{x}}{2\sqrt{x}} - \frac{\sqrt{x}}{2\sqrt{x}}$

$y' = \frac{5}{2\sqrt{x}} - \frac{2\sqrt{x}}{2\sqrt{x}}$

$y' = \frac{5}{2\sqrt{x}} - 1$

f) $y = \frac{x^2 - 2x}{\sqrt{x}}$

$y' = \frac{x^{1/2}(2x - 2) - \frac{1}{2}x^{-1/2}(x^2 - 2x)}{x}$

$y' = \frac{2x^{3/2} \cdot 2x^{1/2} - 2x^{1/2} \cdot 2x^{1/2} - \frac{1}{\cancel{2\sqrt{x}}}(x^2 - 2x) \cdot \cancel{2x^{1/2}}}{x \cdot 2x^{1/2}}$

$y' = \frac{4x^2 - 4x - x^2 + 2x}{2x^{3/2}} = \frac{3x^2 - 2x}{2x^{3/2}}$

$= \frac{x(3x - 2)}{2x^{3/2}}$

$= \frac{x^{-1/2}(3x - 2)}{2}$

$= \boxed{\frac{3x - 2}{2\sqrt{x}}}$

Page 112

⑥ if $y = u^2 - u^3 + 2u^4$ and $u = \frac{x}{2x-1}$ Find $\left.\frac{dy}{dx}\right|_{x=1}$

(i) $\frac{dy}{du} = 2u - 3u^2 + 8u^3$

(ii) $\frac{du}{dx} = \frac{1(2x-1) - 2x}{(2x-1)^2}$ $\quad \cancel{2x} - 1 - \cancel{2x}$

$\frac{du}{dx} = \frac{-1}{(2x-1)^2}$

(iii) Find u

$u = \frac{x}{2x-1}$

$u = \frac{(1)}{2(1)-1}$

$u = 1$

(iv) $\left.\frac{dy}{dx}\right|_{x=1} = \left[\frac{dy}{du}\right] \cdot \left[\frac{du}{dx}\right]$

$\left.\frac{dy}{dx}\right|_{x=1} = \left[2u - 3u^2 + 8u^3\right]\left[\frac{-1}{(2x-1)^2}\right]$

$\left.\frac{dy}{dx}\right|_{x=1} = \left[2(1) - 3(1)^2 + 8(1)^3\right]\left[\frac{-1}{(2(1)-1)^2}\right]$

$\left.\frac{dy}{dx}\right|_{x=1} = \left[2 - 3 + 8\right]\left[-1\right] = -7$

Page 112

(9) d, $y = x\sqrt{x^2+5}$, $(-2,-6)$ $x_1 = -2$ $y_1 = -6$

(1) Find y'

$y' = 1\sqrt{x^2+5} + x\left[\frac{1}{\cancel{2}}(x^2+5)^{-1/2} \cdot \cancel{2}x\right]$

$y' = \sqrt{x^2+5} + \frac{x^2}{\sqrt{x^2+5}}$

(2) Find m or $y'(-2)$

$y'(-2) = \sqrt{(-2)^2+5} + \frac{(-2)^2}{\sqrt{(-2)^2+5}}$

$y' = 3 + \frac{4}{3} = \boxed{\frac{13}{3}}$

(3) Find equation:

$y + 6 = \frac{13}{3}(x + 2)$

$y + 6 = \frac{13x}{3} + \frac{26}{3} - 6$

$\boxed{y = \frac{13x}{3} + \frac{8}{3}}$

$3y = 13x + 8$

$\boxed{0 = 13x - 3y + 8}$

Page 112

(12) Find the points on the curve $y = \frac{1}{2x-1}$ where the tangent line is perpendicular to $x - 2y = 1$

(i) Find perpendicular slope:

$x - 2y = 1$

$-2y = -x + 1$

$y = \frac{-x}{-2} + \frac{1}{-2}$

$y = \frac{1}{2}x - \frac{1}{2}$

$m = \frac{1}{2}$

$\boxed{m\perp = -2}$

(ii) Find derivative:

$y = \frac{1}{2x-1} = (2x-1)^{-1}$

$y' = -1(2x-1)^{-2}(2)$

$\boxed{y' = \frac{-2}{(2x-1)^2}}$

(iii) Solve for x

$\frac{-2}{1} = \frac{-2}{(2x-1)^2}$

$-2(2x-1)^2 = -2$

$(2x-1)^2 = 1$

$4x^2 - 4x + 1 = 1$

$4x^2 - 4x = 0$

$4x(x-1) = 0$

$4x = 0 \quad | \quad x - 1 = 0$

$x = 0 \quad | \quad x = 1$

(iv) Solve for y:

if $x = 0$

$y = \frac{1}{2x-1}$

$y = \frac{1}{-1} = -1$

$\boxed{(0,-1)}$

if $x = 1$

$y = \frac{1}{2x-1}$

$y = \frac{1}{1} = 1$

$\boxed{(1,1)}$

Page 103

③ Find $\left.\frac{dy}{dx}\right]_{x=4}$ if $y=u^2-2u^5$ and $u=x-\sqrt{x}$

① Find u

$u=x-\sqrt{x}$

$u=(4)-\sqrt{4}$

$\boxed{u=2}$

② Find $\frac{dy}{du}$

$y=u^2-2u^5$

$\frac{dy}{du}=2u-10u^4$

③ Find $\frac{du}{dx}$

$u=x-x^{1/2}$

$\frac{du}{dx}=1-\frac{1}{2}x^{-1/2}$

$$\left.\frac{dy}{dx}\right]_{x=4}=(2u-10u^4)\left(1-\frac{1}{2\sqrt{x}}\right)$$

$$=(2(2)-10(2)^4)\left(1-\frac{1}{2\sqrt{4}}\right)$$

$$=(4-160)\left(1-\frac{1}{4}\right)$$

$$=(-156)\left(\frac{3}{4}\right)$$

$$=\frac{-468}{4}$$

$$=-117$$

⑨ If $F(x)=f(g(x))$, where $\boxed{g(2)=4}$, $\boxed{g'(2)=3}$, $\boxed{f'(4)=5}$ find $F'(2)$

$F'(x)=f'(g(x))\cdot g'(x)$

$F'(2)=f'(g(2))\cdot g'(2)$

$F'(2)=\boxed{f'(4)}\cdot\boxed{g'(2)}$

$F'(2)=5\cdot 3=\underline{15}$

If $f(3)=-2$, $f'(3)=3$, $g(3)=1$, $g'(3)=7$

and $f'(1)=4$

Find:

(i) $(f \cdot g)(3) = f'(3)g(3) + f(3) \cdot g'(3)$

$= (3)(1) + (-2)(7)$

$= 3 - 14$

$= -11$

(ii) $\left(\dfrac{f}{g}\right)'(3) = \dfrac{f'(3)g(3) - g'(3)f(3)}{[g(3)]^2}$

$= \dfrac{(3)(1) - (7)(-2)}{(1)^2}$

$= \dfrac{3+14}{1}$

$= 17$

(iii) $(f \circ g)'(3) = f'(g(3)) \cdot g'(3)$

$= f'(1) \cdot g'(3)$

$= 4 \cdot 7$

$= 28$