Arithmetic

$d = t_2 - t_1$

$t_n = a + (n-1)d$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_n = \frac{n}{2}(a + t_n)$

Geometric:

$r = \frac{t_2}{t_1}$

$t_n = ar^{n-1}$

$S_n = \frac{a[r^n - 1]}{[r-1]}$

$S_n = \frac{a}{1-r}$

Infinite Geometric

$-1 < r < 1$

Series + Sequence

① $a = 90000$

$t_6 = 158\ 610.75$

$n = 6$

$r = ?$

$t_n = ar^{n-1}$

$$\frac{158\ 610.75}{90000} = \frac{\cancel{90000}\, r^{6-1}}{\cancel{90000}}$$

$1.7623\ldots = r^5$

$1.12 = r$

$\text{AROI} = 100(1.12 - 1)$

$= 12\%$

Series + Sequence

(2) $a = 105$

$d = 90$

$t_6 = ?$

$t_n = a + (n-1)d$

$t_6 = 105 + (6-1)(90)$

$t_6 = 105 + 5(90)$

$t_6 = 105 + 450$

$t_6 = \$555$

Series + Sequence

3 $a = 5 \times 3$

$a = 15$

$r = 3$

a) Cycle 1 = $a = 15$

b) $t_n = ar^{n-1}$

$t_n = 15(3)^{n-1}$

c) $\frac{32805}{15} = \frac{\cancel{15}(3)^{n-1}}{\cancel{15}}$

$2187 = 3^{n-1}$

$\cancel{3}^7 = \cancel{3}^{n-1}$

$7 = n - 1$

$\boxed{8 = n}$

Cycle 8

d) $S_n = \frac{a(r^n - 1)}{r - 1}$

$S_4 = \frac{15(3^4 - 1)}{3 - 1}$

$S_4 = \frac{15(\cancel{80})}{\cancel{2}}$ 40

$\boxed{S_4 = 600}$

Series + Sequence:

4 a) $\lim_{n\to\infty} \frac{1-2n^4}{3n^4+5} = \frac{-2}{3}$

b) $\lim_{n\to\infty} \frac{n^4-1}{n^6+7} = 0$

c) $\lim_{n\to\infty} \frac{n^2-7n+12}{n+2} = \text{DNE}$

Series + Sequence:

5 a) $\sum_{n=1}^{6} n^2-1 = 0+3+8+15+24+35$

$= 85$

b) $\sum_{n=1}^{\infty}(5)\left(\frac{1}{4}\right)^{n-1}$

infinite geometric

$a=5$
$r=\frac{1}{4}$

$S_n = \frac{a}{1-r}$

$S_n = \frac{5}{1-\frac{1}{4}}$

$S_n = \frac{5}{\frac{3}{4}}$

$S_n = 5 \cdot \frac{4}{3}$

$S_n = \frac{20}{3}$

c) $1+4+7+\ldots+55$

$a=1$
$d=3$
$t_n=55$

(i) Find n:

$t_n = a+(n-1)d$

$55 = 1 + (n-1)3$

$\frac{54}{3} = \frac{3(n-1)}{3}$

$18 = n-1$

$19 = n$

(ii) Find S_{19}

$S_n = \frac{n}{2}(a+t_n)$

$S_{19} = \frac{19}{2}(1+55)$

$S_{19} = \frac{19}{\cancel{2}}(\overset{28}{\cancel{56}})$

$\boxed{S_{19} = 532}$

d) $S_8 = 2-6+18\ldots$

$a=2$
$r=-3$
$n=8$

$S_n = \frac{a(r^n-1)}{r-1}$

$S_8 = \frac{2((-3)^8-1)}{-3-1}$

$S_8 = \frac{2(6560)}{-4}$

$\boxed{S_8 = -3280}$

Series + Sequence:

6. $t_3 = 5$

$t_3 = ar^{3-1}$

$t_3 = ar^2$

$5 = ar^2$

$t_7 = 405$

$t_7 = ar^{7-1}$

$t_7 = ar^6$

$405 = ar^6$

Eliminate "a" by dividing

$$\frac{405 = ar^6}{5 = ar^2}$$

$81 = r^4$

$\underline{\underline{\pm 3 = r}}$

$ar^2 = 5$

$a(3)^2 = 5$

$9a = 5$

$\underline{\underline{a = \frac{5}{9}}}$

$$\boxed{t_n = \frac{5}{9}(-3)^{n-1} \text{ or } t_n = \frac{5}{9}(3)^{n-1}}$$

Series + Sequence:

7. $t_7 = 19$

$t_n = a + (n-1)d$

$t_7 = a + (7-1)d$

$t_7 = a + 6d$

$19 = a + 6d$

$S_{10} = 130$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{10} = \frac{10}{2}[2a + (10-1)d]$

$S_{10} = 5(2a + 9d)$

$S_{10} = 10a + 45d$

$130 = 10a + 45d$

Eliminate "a" by subtracting

$190 = 10a + 60d$

$(-)\ 130 = 10a + 45d$

$\frac{60}{15} = \frac{15d}{15}$

$\boxed{4 = d}$

Find a:

$19 = a + 6d$

$19 = a + 6(4)$

$19 = a + 24$

$\boxed{-5 = a}$

$\boxed{-5, -1, 3, \ldots}$

Limits

① a) $\lim_{x \to 7} \frac{(\sqrt{x+9} - 4)(\sqrt{x+9} + 4)}{(x-7)(\sqrt{x+9} + 4)}$

$\lim_{x \to 7} \frac{x+9-16}{(x-7)(\sqrt{x+9} + 4)}$

$\lim_{x \to 7} \frac{\cancel{(x-7)}}{\cancel{(x-7)}(\sqrt{x+9} + 4)} = \frac{1}{8}$

b) $\lim_{x \to 4} \frac{16 - x^2}{2x^2 - 11x + 12}$ ← Diff of squares ← Hard trinomial

$-8 + -3 = -11$

$-8 \times -3 = 24$

$= \lim_{x \to 4} \frac{(4-x)(4+x)}{(x - \frac{8}{2})(x - \frac{3}{2})}$

$= \lim_{x \to 4} \frac{\cancel{(4-x)}(4+x)}{\cancel{(x-4)}(2x-3)} = \frac{-8}{5}$

Limits

① c) $\lim\limits_{a\to b} \dfrac{(a+2b)^2-9b^2}{a-b}$

$\lim\limits_{a\to b} \dfrac{(a+2b+3b)(a+2b-3b)}{(a-b)}$

$\lim\limits_{a\to b} \dfrac{(a+5b)\cancel{(a-b)}}{\cancel{(a-b)}} = 6b$

d) $\lim\limits_{x\to 0} \dfrac{x^3+1}{x^2+3x+2} = \dfrac{1}{2}$

e) $\lim\limits_{x\to\infty} \dfrac{9-x^4}{(3x^2-2)^2}$

$= \lim\limits_{x\to\infty} \dfrac{9-x^4}{9x^4-12x^2+4}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{9}{x^4}-\frac{x^4}{x^4}}{\frac{9x^4}{x^4}-\frac{12x^2}{x^4}+\frac{4}{x^4}}$

$= \lim\limits_{x\to\infty} \dfrac{9/x^4 - 1}{9 - 12/x^2 + 4/x^4} = -\dfrac{1}{9}$

approaching 0.

Limits

① f) $\lim\limits_{x \to 0} \dfrac{\dfrac{1}{x+5} - \dfrac{1}{5}}{(x^2+5x)(5(x+5))}$ CD: $5(x+5)$

$\lim\limits_{x \to 0} \dfrac{5 - 1(x+5)}{x(x+5)(5(x+5))}$

$\lim\limits_{x \to 0} \dfrac{5-x-5}{5x(x+5)^2}$

$\lim\limits_{x \to 0} \dfrac{-\cancel{x}}{5\cancel{x}(x+5)^2} = \dfrac{-1}{5(25)} = -\dfrac{1}{125}$

g) $\lim\limits_{x \to b} \dfrac{x^2-b^2}{x^8-b^8}$

$\lim\limits_{x \to b} \dfrac{(x^2-b^2)}{(x^4+b^4)(x^4-b^4)}$

$\lim\limits_{x \to b} \dfrac{\cancel{(x^2-b^2)}}{(x^4+b^4)(x^2+b^2)\cancel{(x^2-b^2)}} = \dfrac{1}{(2b^4)(2b^2)} = \dfrac{1}{4b^6}$

h) $\lim\limits_{h \to -4} \dfrac{8+(2+h)^3}{h+4}$ ← sum of cubes

$= \lim\limits_{h \to -4} \dfrac{[2+(2+h)][4-2(2+h)+(2+h)^2]}{h+4}$

$= \lim\limits_{h \to -4} \dfrac{\cancel{(h+4)}[4-2(2+h)+(2+h)^2]}{\cancel{(h+4)}}$

$= 4+4+4$

$= 12$

Limits

2 Let $f(x) = \begin{cases} -x^2+2, & \text{if } x<1 \\ 1, & \text{if } x=1 \\ -2x+3, & \text{if } 1<x\leq 3 \\ -3, & \text{if } x>3 \end{cases}$

$-x^2+2$

x	y
1	1
0	2
-1	1
-2	-2

1

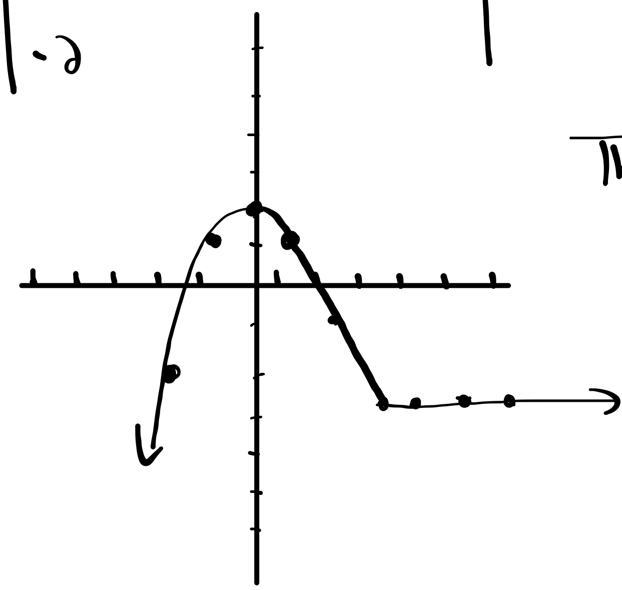
x	y
1	1

$-2x+3$

x	y
1	1
2	-1
3	-3

-3

x	y
3	-3
4	-3
5	-3
6	-3



The function is continuous

Derivatives

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

① a) $f(x) = \sqrt{x-5}$

(i) $f(x+h) = \sqrt{x+h-5}$

(ii) $$f'(x) = \lim_{h\to 0} \frac{\left(\sqrt{x+h-5} - \sqrt{x-5}\right)\left(\sqrt{x+h-5} + \sqrt{x-5}\right)}{h\left(\sqrt{x+h-5} + \sqrt{x-5}\right)}$$

$$f'(x) = \lim_{h\to 0} \frac{x+h-5-(x-5)}{h\left(\sqrt{x+h-5} + \sqrt{x-5}\right)}$$

$$f'(x) = \lim_{h\to 0} \frac{x+h-5-x+5}{h\left(\sqrt{x+h-5} + \sqrt{x-5}\right)}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+\underline{h}-5} + \sqrt{x-5}\right)}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x-5}}}$$

Derivatives

① b) $f(x) = 3x^2 - 5x + 1$

$$f(x+h) = 3(x+h)^2 - 5(x+h) + 1$$
$$= 3(x^2 + 2xh + h^2) - 5x - 5h + 1$$
$$= 3x^2 + 6xh + 3h^2 - 5x - 5h + 1$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{5x} - 5h + \cancel{1} - \cancel{3x^2} + \cancel{5x} - \cancel{1}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{6xh + 3h^2 - 5h}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{h}(6x + 3h - 5)}{\cancel{h}}$$

$$\boxed{f'(x) = 6x - 5}$$

Derivatives

2 a) $f(x) = 3x^2 + 5x - 2$

$\boxed{f'(x) = 6x + 5}$

b) $f(x) = \frac{3}{\sqrt{x}}$

$f(x) = 3x^{-1/2}$

$f'(x) = -\frac{3}{2}x^{-3/2}$

$\boxed{f'(x) = -\frac{3}{2x^{3/2}}}$

c) $f(x) = 2x^4 + \sqrt{x}$

$f(x) = 2x^4 + x^{1/2}$

$f'(x) = 8x^3 + \frac{1}{2}x^{-1/2}$

$\boxed{f'(x) = 8x^3 + \frac{1}{2\sqrt{x}}}$

d) $f(x) = \sqrt[3]{x^2}$

$f(x) = x^{2/3}$

$f'(x) = \frac{2}{3}x^{-1/3}$

$\boxed{f'(x) = \frac{2}{3x^{1/3}}}$

Derivatives

3 a) $y = (3x^2-2)(4x+5)$

$$\frac{dy}{dx} = (6x)(4x+5) + 4(3x^2-2)$$

$$\frac{dy}{dx} = 24x^2+30x+12x^2-8$$

$$\boxed{\frac{dy}{dx} = 36x^2+30x-8}$$

b) $g(x) = (x^2-5x+2)(4x+1)$

$$g'(x) = (2x-5)(4x+1) + 4(x^2-5x+2)$$

$$g'(x) = 8x^2+2x-20x-5+4x^2-20x+8$$

$$\boxed{g'(x) = 12x^2-38x+3}$$

Derivatives

4 a) $f(x) = \dfrac{2x^2+3}{3x-2}$

$$f'(x) = \frac{4x(3x-2) - 3(2x^2+3)}{(3x-2)^2}$$

$$f'(x) = \frac{12x^2-8x-6x^2-9}{(3x-2)^2}$$

$$\boxed{f'(x) = \frac{6x^2-8x-9}{(3x-2)^2}}$$

b) $y = \dfrac{\sqrt{x}}{3+x^2}$

$$y' = \frac{\frac{1}{2}x^{-1/2}(3+x^2) - 2x(x^{1/2})}{(3+x^2)^2}$$

$$y' = \frac{\frac{3+x^2}{\cancel{2x^{1/2}}} \cdot \cancel{2x^{1/2}} - 2x^{3/2} \cdot 2x^{1/2}}{(3+x^2)^2 \cdot 2x^{1/2}}$$

$$y' = \frac{3+x^2-4x^2}{2x^{1/2}(3+x^2)^2}$$

$$\boxed{y' = \frac{3-3x^2}{2\sqrt{x}(3+x^2)^2}}$$

Derivatives

5) $y = (x^2 - 3)^8$ @ $x = \underline{\underline{2}}$

(i) Find y:

$y = (2^2 - 3)^8$

$y = (4-3)^8$

$y = \underline{\underline{1}}$

(ii) Find y':

$y = (x^2 - 3)^8$

$y' = 8(x^2-3)^7(2x)$

$y' = 16x(x^2-3)^7$

(iii) Find slope (m):

$y' = 16(2)[(2)^2 - 3]^7$

$y' = 32[4-3]^7$

$y' = 32 \leftarrow m = 32$

(iv): $y - y_1 = m(x - x_1)$

$y - 1 = 32(x - 2)$

$y - 1 = 32x - 64$

$\boxed{y = 32x - 63}$

or $\boxed{32x - y - 63 = 0}$

7) a) $f(x) = \left(\frac{2x+1}{x-1}\right)^5$

$$f'(x) = 5\left(\frac{2x+1}{x-1}\right)^4 \left[\frac{\overbrace{2(x-1)}^{2x-2} - \overbrace{1(2x+1)}^{-2x-1}}{(x-1)^2}\right]$$

$$f'(x) = 5\frac{(2x+1)^4}{(x-1)^4} \cdot \frac{-3}{(x-1)^2}$$

$$f'(x) = \frac{-15(2x+1)^4}{(x-1)^6}$$

Derivatives

6 a) $f(x) = 3(2x^2-4)^4$

$f'(x) = 12(2x^2-4)^3(4x)$

$\boxed{f'(x) = 48x(2x^2-4)^3}$

b) $y = \dfrac{16}{\sqrt{x-1}}$

$y = 16(x-1)^{-1/2}$

$\dfrac{dy}{dx} = -8(x-1)^{-3/2}(1)$

$\dfrac{dy}{dx} = \dfrac{-8}{(x-1)^{3/2}}$

or $\boxed{\dfrac{dy}{dx} = \dfrac{-8}{\sqrt{(x-1)^3}}}$

Derivatives

7) a) $f(x) = \left(\dfrac{2x+1}{x-1}\right)^5$

$$f'(x) = 5\left[\frac{2x+1}{x-1}\right]^4 \left[\frac{\overbrace{2(x-1)}^{2x-2} - \overbrace{1(2x+1)}^{-2x-1}}{(x-1)^2}\right]$$

$$f'(x) = 5 \cdot \frac{(2x+1)^4}{(x-1)^4} \cdot \frac{-3}{(x-1)^2}$$

$$\boxed{f'(x) = \frac{-15(2x+1)^4}{(x-1)^6}}$$

Derivatives

7 b) $y = (x^2-1)^3(3x-2)^2$

$$\frac{dy}{dx} = 3(x^2-1)^2(2x)(3x-2)^2 + (x^2-1)^3(2)(3x-2)(3)$$

$$\frac{dy}{dx} = 6x(x^2-1)^2(3x-2)^2 + 6(x^2-1)^3(3x-2)$$

$3x^2 - 2x + x^2 - 1$

$$\frac{dy}{dx} = 6(x^2-1)^2(3x-2)\left[x(3x-2) + (x^2-1)\right]$$

$$\boxed{\frac{dy}{dx} = 6(x^2-1)^2(3x-2)(4x^2-2x-1)}$$

Curve Sketching:

① Plot all points: x-int, y-int, max, min, I.P.,

② Plot all asymptotes (Check behaviour near VA)

③ Use intervals of inc/dec and concavity to connect everything

Curve Sketching

② $f(x) = x^4 - 3x^3 + 3x^2 - x$

$f(x) = x(\underbrace{x^3 - 3x^2 + 3x - 1})$

Synthetic Substitution:

Find an x-value that makes $f(x) = 0$

$f(1) = (1)^3 - 3(1)^2 + 3(1) - (1)$

$f(1) = 1 - 3 + 3 - 1$

$f(1) = 0$ ✓

$x = 1$

$x - 1 = 0$

② $1|$

1	-3	3	-1
	1	-2	1
1	-2	1	$\varnothing$

$x\,(x-1)(x^2 - 2x + 1)$

common factor

simple trinomial

$f(x) = x(x-1)(x-1)(x-1)$

$f(x) = x(x-1)^3$

Sketch $f(x) = x^3 - 12x^2 + 36x$

$f'(x) = 3x^2 - 24x + 36$

$f''(x) = 6x - 24$

① x-int

$0 = x^3 - 12x^2 + 36x$

$0 = x(x^2 - 12x + 36)$

$0 = x(x-6)(x-6)$

$x = 0 \quad | \quad x - 6 = 0$

$\qquad\quad\; x = 6$

$(0,0) \quad (6,0)$

② y-int $(x=0)$

$y = (0)^3 - 12(0)^2 + 36(0)$

$y = 0$

$(0,0)$

③ Intervals of Inc/Dec

$f'(x) = 3x^2 - 24x + 36$

$f'(x) = 3(x^2 - 8x + 12)$

$f'(x) = 3(x-6)(x-2)$

$0 = 3(x-6)(x-2)$

CV: $x = 6$, $x = 2$

max min

$+ \quad - \quad +$

(1) 2 (3) 6 (7)

Increasing on $(-\infty, 2) + (6, \infty)$

Decreasing on $(2, 6)$

④ max/min:

max @ $x = 2$

$f(x) = x^3 - 12x^2 + 36x$

$f(2) = (2)^3 - 12(2)^2 + 36(2)$

$f(2) = 8 - 48 + 72$

$f(2) = 32$

$(2, 32)$

min @ $x = 6$

$f(x) = x^3 - 12x^2 + 36x$

$f(6) = (6)^3 - 12(6)^2 + 36(6)$

$f(6) = 216 - 432 + 216$

$f(6) = 0$

$(6, 0)$

⑤ Intervals of Concavity.

$f''(x) = 6x - 24$

$f''(x) = 6(x-4)$

$0 = 6(x-4)$

CV: $x = 4$

IP

$- \quad +$

(0) 4 (5)

CD on $(-\infty, 4)$

CU on $(4, \infty)$

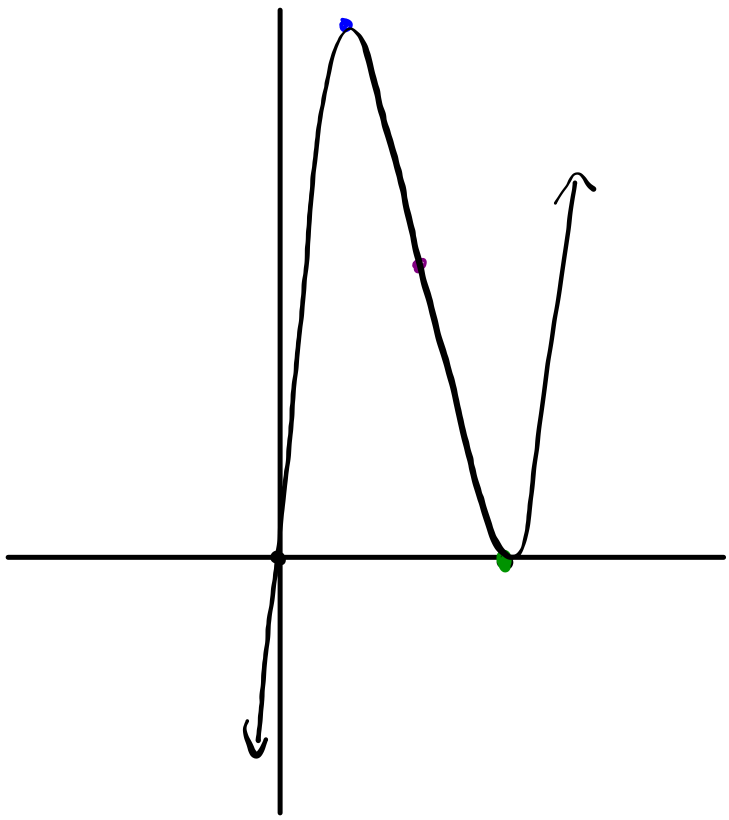
⑥ Inflection Point @ $x = 4$

$f(x) = x^3 - 12x^2 + 36x$

$f(4) = (4)^3 - 12(4)^2 + 36(4)$

$f(4) = 64 - 192 + 144$

$f(4) = 16$



Curve Sketching

Example:

Examine the function $f(x) = 3x^5 - 5x^3$ with respect to...

- Intercepts $f(x)$
- Symmetry
- Asymptotes (No asymptotes for polynomial functions)
- Intervals of Increase or Decrease $f'(x)$
- Local Maximum and Minimum values $f(x)$
- $f''(x)$ Concavity and Points of Inflection $f(x)$
- Sketch the Curve

$f(x) = 3x^5 - 5x^3$ $\quad$ $f'(x) = 15x^4 - 15x^2$ $\quad$ $f''(x) = 60x^3 - 30x$

$f(x) = x^3(3x^2 - 5)$ $\quad$ $f'(x) = 15x^2(x^2 - 1)$ $\quad$ $f''(x) = 30x(2x^2 - 1)$

$f'(x) = 15x^2(x-1)(x+1)$

① x-int (y=0)

$f(x) = x^3(3x^2-5)$

$0 = x^3(3x^2-5)$

$x^3 = 0$ | $3x^2 - 5 = 0$

$x = 0$ | $\frac{3x^2}{3} = \frac{5}{3}$

(0,0) | $x^2 = \frac{5}{3}$

$x = \pm\sqrt{5/3}$

(1.29, 0)

(-1.29, 0)

② y-int (x=0)

$f(x) = 3x^5 - 5x^3$

$f(0) = 3(0)^5 - 5(0)^3$

$f(0) = 0$

(0,0)

Symmetry

$f(x) = 3x^5 - 5x^3$

$f(-x) = 3(-x)^5 - 5(-x)^3$

$f(x) = -3x^5 + 5x^3$

$f(x) = -(3x^5 - 5x^3)$

Since $f(-x) = -f(x)$ the function is Odd

③ Intervals of Inc/Dec.

$f'(x) = 15x^2(x-1)(x+1)$

$0 = 15x^2(x-1)(x+1)$

$15x^2 = 0$ | $x - 1 = 0$ | $x + 1 = 0$

$x^2 = 0$ | $x = 1$ | $x = -1$

$x = 0$

CV: x = -1, 0, 1

max $\quad$ min

$+ \quad - \quad - \quad +$

(-2) -1 (-½) 0 (½) 1 (2)

Increasing on $(-\infty, -1)$ & $(1, \infty)$

Decreasing on $(-1, 0)$ & $(0, 1)$

or $(-1, 1)$

④ max @ <u>x = -1</u>

$f(x) = 3x^5 - 5x^3$

$f(-1) = 3(-1)^5 - 5(-1)^3$

$f(-1) = -3 + 5$

$f(-1) = 2$

(-1, 2)

⑤ min @ <u>x = 1</u>

$f(x) = 3x^5 - 5x^3$

$f(1) = 3(1)^5 - 5(1)^3$

$f(1) = 3 - 5$

$f(1) = -2$

(1, -2)

⑥ Intervals of Concavity:

$f''(x) = 30x(2x^2 - 1)$

$0 = 30x(2x^2 - 1)$

$30x = 0$ | $2x^2 - 1 = 0$

$x = 0$ | $2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{1/2}$

$x \approx \pm 0.707$

CV: x = -0.7, 0, 0.7

I.P. $\quad$ I.P. $\quad$ I.P.

$- \quad + \quad - \quad +$

(-1) -0.7 (-0.5) 0 (0.5) 0.7 (1)

CD on $(-\infty, -0.7)$ + $(0, 0.7)$

CU on $(-0.7, 0)$ + $(0.7, \infty)$

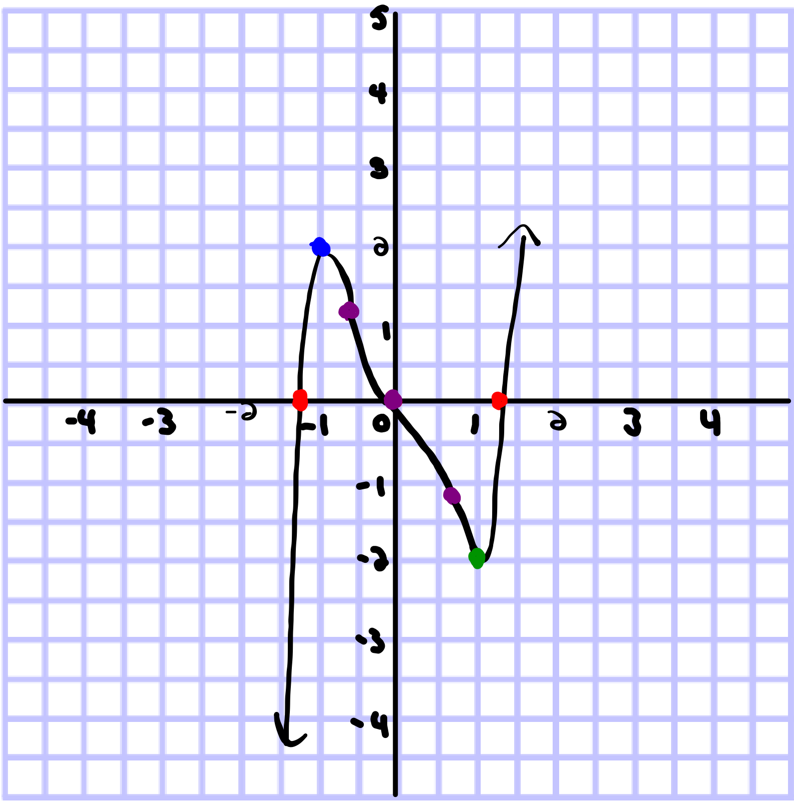
⑦ Inflection Points

$f(x) = 3x^5 - 5x^3$

$f(-0.7) = 3(-0.7)^5 - 5(-0.7)^3 = -0.504 + 1.715 = 1.2$ $\quad$ (-0.7, 1.2)

$f(0) = 3(0)^5 - 5(0)^3 = 0 - 0 = 0$ $\quad$ (0, 0)

$f(0.7) = 3(0.7)^5 - 5(0.7)^3 = 0.504 - 1.715 = -1.2$ $\quad$ (0.7, -1.2)


5
4
3
2
1
-4 -3 -2 -1 0 1 2 3 4
-1
-2
-3
-4

Curve Sketching

Examine the function $f(x)=\frac{x^2}{1-x^2}$ with respect to… $f'(x)=\frac{2x}{(1-x^2)^2}$

- Intercepts $f(x)$
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

$f''(x)=\frac{2(1+3x^2)}{(1-x^2)^3}$

① x-int (y=0)

$f(x)=\frac{x^2}{1-x^2}$

$0=\frac{x^2}{1-x^2}$

$0=x^2$

$0=x$

(0,0)

② y-int (x=0)

$f(x)=\frac{x^2}{1-x^2}$

$f(0)=\frac{(0)^2}{1-(0)^2}$

$f(0)=\frac{0}{1}=0$

(0,0)

Symmetry:

$f(x)=\frac{x^2}{1-x^2}$

$f(-x)=\frac{(-x)^2}{1-(-x)^2}$

$f(-x)=\frac{x^2}{1-x^2}$

Since $f(-x)=f(x)$ the function is f

③ Vertical Asymptote:
(zeroes of the denominator)

$f(x)=\frac{x^2}{1-x^2}$

VA: $1-x^2=0$

$(1-x)(1+x)=0$

$1-x=0$ | $1+x=0$

$\boxed{1=x}$ | $\boxed{x=-1}$

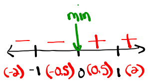
④ Horizontal Asymptote:

$\lim_{x\to\infty}\frac{x^2}{1-x^2}=\frac{1}{-1}=-1$

$\boxed{y=-1}$

⑤ Intervals of Inc/Dec.

$f'(x)=\frac{2x}{(1-x^2)^2}$



$2x=0$ | $(1-x^2)^2=0$

$x=0$ | $1-x^2=0$

$1=x^2$

$\pm1=x$

CV: $x=-1,0,1$

Increasing on $(0,\infty)$

Decreasing on $(-\infty,0)$

⑥ min @ $\underline{x=0}$

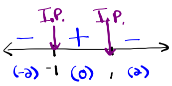
$f(x)=\frac{x^2}{1-x^2}$

$f(0)=\frac{(0)^2}{1-(0)^2}=0$

(0,0)

⑦ Intervals of Concavity

$f''(x)=\frac{2(1+3x^2)}{(1-x^2)^3}$



$2(1+3x^2)=0$ | $(1-x^2)^3=0$

$1+3x^2=0$ | $1-x^2=0$

$3x^2=-1$ | $1=x^2$

$x^2=-\frac{1}{3}$ | $\pm1=x$

Not Possible

(Numerator is always positive)

CD on $(-\infty,-1)+(1,\infty)$

CU on $(-1,1)$

⑧ Inflection Points:

when $\underline{x=-1}$

$f(x)=\frac{x^2}{1-x^2}$

$f(-1)=\frac{(-1)^2}{1-(-1)^2}=\frac{1}{0}=$ und.

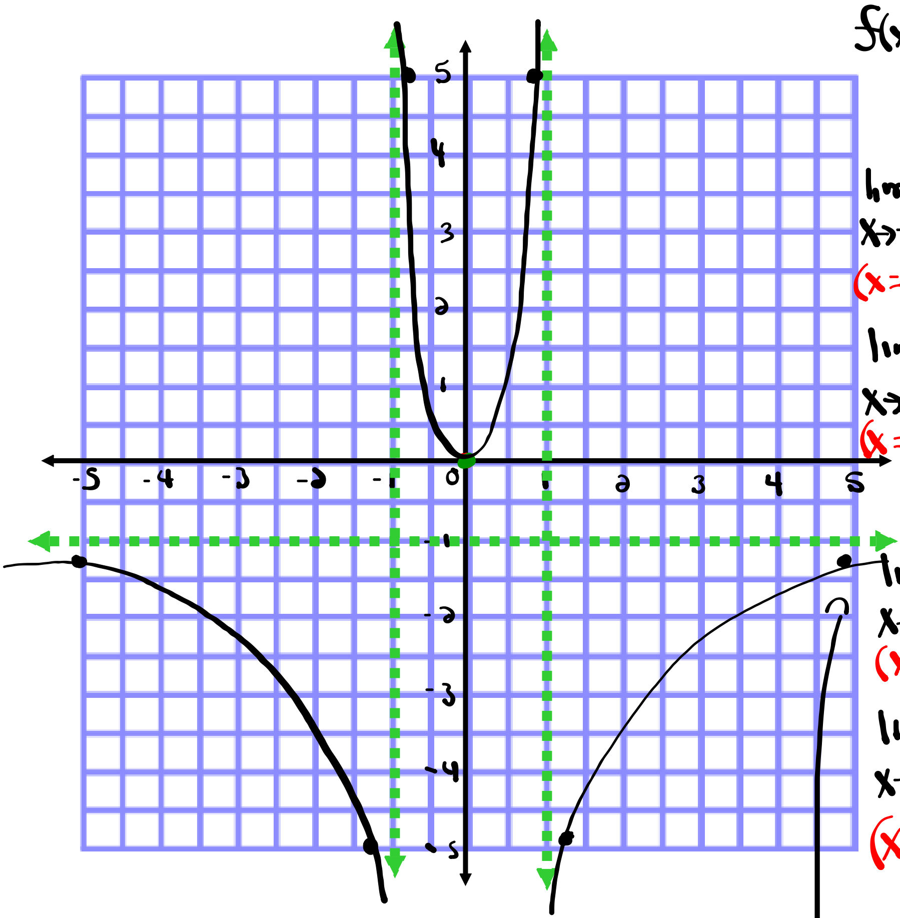
when $\underline{x=1}$

$f(x)=\frac{x^2}{1-x^2}$

$f(1)=\frac{(1)^2}{1-(1)^2}=\frac{1}{0}=$ u

$f(x)=\frac{x^2}{1-x^2}$

$f(x)=\frac{x^2}{(1-x)(1+x)}$



$\lim_{x\to -1^-} f(x) = \frac{(+)}{(+)(-)} = -\infty$

(x=-1.1)

$\lim_{x\to -1^+} f(x) = \frac{(+)}{(+)(+)} = +\infty$

(x=-0.9)

$\lim_{x\to 1^-} f(x) = \frac{(+)}{(+)(+)} = +\infty$

(x=0.9)

$\lim_{x\to 1^+} f(x) = \frac{(+)}{(-)(+)} = -\infty$

(x=1.1)

$f(5)=\frac{(5)^2}{1-(5)^2}=\frac{25}{-24}$

Permutations + Combinations

binomial theore

- used to expand $(x + y)^n$, $n \in N$
- each term has the f $_nC_k(x)^{n-k}(y)^k$, where $k + 1$ is the term number

You can use the **binomial theorem** to expand any power of a binomial expression.

$$(x + y)^n = {}_nC_0(x)^n(y)^0 + {}_nC_1(x)^{n-1}(y)^1 + {}_nC_2(x)^{n-2}(y)^2 + \cdots + {}_nC_{n-1}(x)^1(y)^{n-1} + {}_nC_n(x)^0(y)^n$$

In this chapter, all binomial expansions will be written in descending order of the exponent of the first term in the binomial.

The following are some important observations about the expansion of $(x + y)^n$, where x and y represent the terms of the binomial and $n \in N$:

- the expansion contains $n + 1$ terms
- the number of objects, k, selected in the combination $_nC_k$ can be taken to match the number of factors of the second variable selected; that is, it is the same as the exponent on the second variable
- the general term, t_{k+1}, has the form

$$_nC_k(x)^{n-k}(y)^k$$

the same

- the sum of the exponents in any term of the expansion is n

Permutations + Combinations

4. $(2x - 5y^2)^{11}$ has 12 terms. B

5. Third term of $(2x^2 + 3y)^7$ expanded is

$$_7C_2 (2x^2)^5 (3y)^2$$

$$21(32x^{10})(9y^2)$$

$$\boxed{6048x^{10}y^2}$$ A

Permutations + Combinations

12 $\left(x^2 + \frac{2}{x}\right)^9$ $x^{18} \ldots x^{15} \ldots x^{12} \ldots x^9$

$_9C_3 (x^2)^6 \left(\frac{2}{x}\right)^3$

$84 \left(x^{12}\right)\left(\frac{8}{x^3}\right)$

$\frac{672x^{12}}{x^3}$

$\boxed{672x^9}$ the 4th term

Permutations + Combinations

15 $\left(y - \frac{2}{y^2}\right)^5$

$$ {}_5C_0(y)^5\left(\frac{-2}{y^2}\right)^0 + {}_5C_1(y)^4\left(\frac{-2}{y^2}\right)^1 + {}_5C_2(y)^3\left(\frac{-2}{y^2}\right)^2 + {}_5C_3(y)^2\left(\frac{-2}{y^2}\right)^3 + {}_5C_4(y)^1\left(\frac{-2}{y^2}\right)^4 + {}_5C_5(y)^0\left(\frac{-2}{y^2}\right)^5 $$

$$ 1(y^5)(1) + 5(y^4)\left(\frac{-2}{y^2}\right) + 10(y^3)\left(\frac{4}{y^4}\right) + 10(y^2)\left(\frac{-8}{y^6}\right) + 5(y)\left(\frac{16}{y^8}\right) + 1(1)\left(\frac{-32}{y^{10}}\right) $$

$$ y^5 - \frac{10y^4}{y^2} + \frac{40y^3}{y^4} - \frac{80y^2}{y^6} + \frac{80y}{y^8} - \frac{32}{y^{10}} $$

$$ y^5 - 10y^2 + \frac{40}{y} - \frac{80}{y^4} + \frac{80}{y^7} - \frac{32}{y^{10}} $$

Review Composite Functions

② Suppose $f(x) = x^2 - 3x + 5$ and $g(x) = 2x - 3$

composed with

a) find $(f \circ g)(x)$

$f(g(x)) = (g(x))^2 - 3(g(x)) + 5$

$f(2x-3) = (2x-3)^2 - 3(2x-3) + 5$

$f(2x-3) = 4x^2 - 12x + 9 - 6x + 9 + 5$

$f(2x-3) = 4x^2 - 18x + 23$

b) find $g(f(x))$

$g(f(x)) = 2(f(x)) - 3$

$g(x^2-3x+5) = 2(x^2-3x+5) - 3$

$g(x^2-3x+5) = 2x^2 - 6x + 10 - 3$

$g(x^2-3x+5) = 2x^2 - 6x + 7$

c) find $f(g(3))$

(i) $g(3) = 2(3) - 3$

$g(3) = 6 - 3$

$g(3) = 3$

(ii) $f(3) = (3)^2 - 3(3) + 5$

$f(3) = 9 - 9 + 5$

$\boxed{f(3) = 5}$

d) find $g(f(-1))$

(i) $f(-1) = (-1)^2 - 3(-1) + 5$

$f(-1) = 1 + 3 + 5$

$f(-1) = 9$

(ii) $g(9) = 2(9) - 3$

$g(9) = 18 - 3$

$\boxed{g(9) = 15}$

Review Combining Functions

③ Suppose: $f(x) = (x+2)^2 - 3$, $g(x) = 3x+1$, $h(x) = \sqrt{x+5}$, $i(x) = \log(x-3)$

multiply

a) find $(f \cdot g)(x)$ and state its domain.

b) find $(h-g)(x)$ and state its domain.

c) find $\left(\frac{f}{h}\right)(x)$ and state its domain.

d) find $(f+i)(x)$ and state its domain.

$f(x) = (x+2)^2 - 3$	$g(x) = 3x+1$	$h(x) = \sqrt{x+5}$	$i(x) = \log(x-3)$
$f(x) = x^2+4x+4-3$	(degree 1)	(radical)	(logrithm)
$f(x) = x^2+4x+1$	D: $\{x \mid x \in R\}$	$x+5 \geq 0$	$x-3>0$
(degree 2)	or $(-\infty, \infty)$	$x \geq -5$	$x>3$
D: $\{x \mid x \in R\}$		D: $\{x \mid x \geq -5, x \in R\}$	D: $\{x \mid x>3, x \in R\}$
or $(-\infty, \infty)$		or $[-5, \infty)$	or $(3, \infty)$

a) $(f \cdot g)(x) = f(x)g(x)$

$(f \cdot g)(x) = (x^2+4x+1)(3x+1)$

$(f \cdot g)(x) = 3x^3 + x^2 + 12x^2 + 4x + 3x + 1$

$(f \cdot g)(x) = 3x^3 + 13x^2 + 7x + 1$ (degree 3)

D: $\{x \mid x \in R\}$ or $(-\infty, \infty)$

b) $(h-g)(x) = h(x) - g(x)$

$(h-g)(x) = \sqrt{x+5} - (3x+1)$

$(h-g)(x) = \sqrt{x+5} - 3x - 1$

D: $\{x \mid x \geq -5, x \in R\}$ or $[-5, \infty)$

c) $\left(\frac{f}{h}\right)(x) = \frac{f(x)}{h(x)}$

$\left(\frac{f}{h}\right)(x) = \frac{x^2+4x+1}{\sqrt{x+5}} \rightarrow \sqrt{x+5} \neq 0$

$x+5 \neq 0$

$x \neq -5$

D: $\{x \mid x > -5, x \in R\}$

or $(-5, \infty)$

d) $(f+i)(x) = f(x) + i(x)$

$(f+i)(x) = (x+2)^2 - 3 + \log(x-3)$

$x-3>0$

$x>3$

D: $\{x \mid x>3, x \in R\}$

or $(3, \infty)$

Extra Problems Worked Out

Find the term in the expansion of

$(2x+y)^8$ that contains x^5y^3

$_8C_3(2x)^5(y)^3$

$(56)(32x^5)(y^3)$

$1792x^5y^3$

Permutations + Combinations

5 $(2x^2+3y)^7 \rightarrow 3^{rd}$ term

$_7C_2(2x^2)^5(3y)^2$

$21(32x^{10})(9y^2)$

$\boxed{6048x^{10}y^2}$

Permutations + Combinations

② $\left(x^2 - \frac{x}{2}\right)^5$

$${}_5C_0(x^2)^5\left(-\frac{x}{2}\right)^0 + {}_5C_1(x^2)^4\left(-\frac{x}{2}\right)^1 + {}_5C_2(x^2)^3\left(-\frac{x}{2}\right)^2 + {}_5C_3(x^2)^2\left(-\frac{x}{2}\right)^3 + {}_5C_4(x^2)^1\left(-\frac{x}{2}\right)^4 + {}_5C_5(x^2)^0\left(-\frac{x}{2}\right)^5$$

$$1(x^{10})(1) + 5(x^8)\left(-\frac{x}{2}\right) + 10(x^6)\left(\frac{x^2}{4}\right) + 10(x^4)\left(-\frac{x^3}{8}\right) + 5(x^2)\left(\frac{x^4}{16}\right) + 1(1)\left(-\frac{x^5}{32}\right)$$

$$x^{10} - \frac{5x^9}{2} + \frac{5}{2}x^8 - \frac{5}{4}x^7 + \frac{5x^6}{16} - \frac{x^5}{32}$$

Perm/Comb

(2) $\left(x^2 - \frac{x}{3}\right)^3$ $a = x^2$ $b = -\frac{x}{3}$ $n = 3$

$${}_3C_0(x^2)^3\left(-\frac{x}{3}\right)^0 + {}_3C_1(x^2)^2\left(-\frac{x}{3}\right)^1 + {}_3C_2(x^2)^1\left(-\frac{x}{3}\right)^2 + {}_3C_3(x^2)^0\left(-\frac{x}{3}\right)^3$$

$$1(x^6)(1) + 3(x^4)\left(-\frac{x}{3}\right) + 3(x^2)\left(\frac{x^2}{9}\right) + 1(1)\left(-\frac{x^3}{27}\right)$$

$$x^6 - \frac{3x^5}{3} + \frac{3x^4}{9} - \frac{1x^3}{27}$$

$$\boxed{x^6 - x^5 + \frac{1}{3}x^4 - \frac{1}{27}x^3}$$

Permutations + Combinations

Expand: $\left(x - \frac{1}{2}\right)^{4}$

$${}_4C_0(x)^4\left(-\frac{1}{2}\right)^0 + {}_4C_1(x)^3\left(-\frac{1}{2}\right)^1 + {}_4C_2(x)^2\left(-\frac{1}{2}\right)^2 + {}_4C_3(x)^1\left(-\frac{1}{2}\right)^3 + {}_4C_4(x)^0\left(-\frac{1}{2}\right)^4$$

$$(1)(x^4)(1) + (4)(x^3)\left(-\frac{1}{2}\right) + (6)(x^2)\left(\frac{1}{4}\right) + (4)(x)\left(-\frac{1}{8}\right) + (1)(1)\left(\frac{1}{16}\right)$$

$$x^4 - \frac{4x^3}{2} + \frac{6x^2}{4} - \frac{4x}{8} + \frac{1}{16}$$

$$\boxed{x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{16}}$$

$$\lim_{x \to -\infty} \frac{\sqrt{4x^2 - x}}{x+1}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{x^2\left(4 - \frac{1}{x}\right)}}{x\left(1 + \frac{1}{x}\right)}$$

$$= \lim_{x \to -\infty} \frac{|x|\sqrt{4 - \frac{1}{x}}}{x\left(1 + \frac{1}{x}\right)} \qquad |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

$$= \lim_{x \to -\infty} \frac{\overset{-1}{\cancel{-x}}\sqrt{4 - \frac{1}{x}}}{\overset{1}{\cancel{x}}\left(1 + \frac{1}{x}\right)}$$

$$= -2$$

Limits

$$\lim_{x\to 0} \frac{\cancel{3(x+3)}\frac{1}{\cancel{x+3}} - \frac{1}{\cancel{3}}\,\cancel{3}(x+3)}{x\;3(x+3)}$$

CD: $3(x+3)$

$$\lim_{x\to 0} \frac{3-(x+3)}{3x(x+3)}$$

$$\lim_{x\to 0} \frac{3-x-3}{3x(x+3)}$$

$$\lim_{x\to 0} \frac{-\cancel{x}}{3\cancel{x}(x+3)} = \frac{-1}{3(3)} = -\frac{1}{9}$$

Differentiation

Product: $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$

$$y = \underbrace{(3x^2-5)}_{f(x)}\underbrace{(4x^3+3x)}_{g(x)}$$

$$y' = 6x(4x^3+3x) + (3x^2-5)(12x^2+3)$$

$$y' = 24x^4 + 18x^2 + 36x^4 + 9x^2 - 60x^2 - 15$$

$$\boxed{y' = 60x^4 - 33x^2 - 15}$$

Quotient: $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

$$y = \frac{x^2+2 \leftarrow f(x)}{3x+5 \leftarrow g(x)}$$

$$y' = \frac{2x(3x+5) - 3(x^2+2)}{(3x+5)^2}$$

$$y' = \frac{6x^2+10x-3x^2-6}{(3x+5)^2} = \boxed{\frac{3x^2+10x-6}{(3x+5)^2}}$$

Chain: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

$$y = \sqrt{4x^3-6x} = (4x^3-6x)^{1/2}$$

$$y' = \frac{1}{2}(4x^3-6x)^{-1/2}(12x^2-6)$$

$$y' = \frac{12x^2-6}{2(4x^3-6x)^{1/2}} = \frac{\overset{3}{\cancel{6}}(2x^2-1)}{\cancel{2}\sqrt{4x^3-6x}} = \boxed{\frac{3(2x^2-1)}{\sqrt{4x^3-6x}}}$$

$$\frac{6x^2-3}{\sqrt{4x^3-6x}}$$

Combo:

$$y = (3x^2+5)^3\sqrt{4x-2} = \underbrace{(3x^2+5)^3}_{f(x)}\underbrace{(4x-2)^{1/2}}_{g(x)}$$

$$y' = \underbrace{3(3x^2+5)^2(6x)}_{f'(x)}\underbrace{(4x-2)^{1/2}}_{g(x)} + \underbrace{(3x^2+5)^3}_{f(x)}\underbrace{\left(\frac{1}{2}\right)(4x-2)^{-1/2}(4)}_{g'(x)}$$

$$y' = 18x(3x^2+5)^2(4x-2)^{1/2} + 2(3x^2+5)^3(4x-2)^{-1/2}$$

$\cdot\ 36x^2 - 18x + 3x^2 + 5$

$$y' = 2(3x^2+5)^2(4x-2)^{-1/2}\left[9x(4x-2) + (3x^2+5)\right]$$

$$y' = 2(3x^2+5)^2(4x-2)^{-1/2}(39x^2-18x+5)$$

$$y' = \frac{2(3x^2+5)^2(39x^2-18x+5)}{\sqrt{4x-2}}$$

$$\frac{(3x^2+5)^2}{(3x^2+5)^2} = (3x^2+5)^0 = 1 \quad \Bigg| \quad \frac{(4x-2)^{1/2}}{(4x-2)^{-1/2}} = (4x-2)^{1/2-(-1/2)} = (4x-2)^1$$

Differentiation

$$f(x) = \frac{(3x^2+5)^3}{\sqrt{2x-7}} \qquad \frac{f'g - fg'}{g^2}$$

$$f'(x) = \frac{\overbrace{3(3x^2+5)^2(6x)}^{f'}\overbrace{(2x-7)^{\frac{1}{2}}}^{g} - \overbrace{(3x^2+5)^3}^{f}\overbrace{\left(\tfrac{1}{2}\right)(2x-7)^{-\frac{1}{2}}(2)}^{g'}}{\left[\sqrt{2x-7}\right]^2}$$

$$f'(x) = \frac{18x(3x^2+5)^2(2x-7)^{\frac{1}{2}} - (3x^2+5)^3(2x-7)^{-\frac{1}{2}}}{2x-7}$$ Factor

$36x^2 - 126x - 3x^2 - 5$

$$f'(x) = \frac{(3x^2+5)^2(2x-7)^{-\frac{1}{2}}\left[18x(2x-7) - (3x^2+5)\right]}{(2x-7)}$$

$$f'(x) = \frac{(3x^2+5)^2(33x^2-126x-5)}{(2x-7)^{3/2}}$$

Differentiation

③ $$y = \sqrt[7]{2x^2 + \sqrt{x^7 - 8x\sqrt{3-x}}} = \left[2x^2 + \left(x^7 - 8x(3-x)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]^{\frac{1}{7}}$$

$$y' = \frac{1}{7}\left[2x^2 + \left(x^7 - 8x(3-x)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]^{-\frac{6}{7}}\left[4x + \frac{1}{2}\left(x^7 - 8x(3-x)^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(7x^6 - \left(8(3-x)^{\frac{1}{2}} + 8x\left(\frac{1}{2}\right)(3-x)^{-\frac{1}{2}}(-1)\right)\right)\right]$$

(x,y)

Find the point on the curve $y = x^2 + 5x$ where the slope of the tangent line equals 9

(i) Find derivative

$y = x^2 + 5x$

$y' = 2x + 5$

(ii) Solve for x:

$y' = 2x + 5$

$9 = 2x + 5$

$4 = 2x$

$2 = x$

(iii) Solve for y:

$y = x^2 + 5x$

$y = (2)^2 + 5(2)$

$y = 4 + 10$

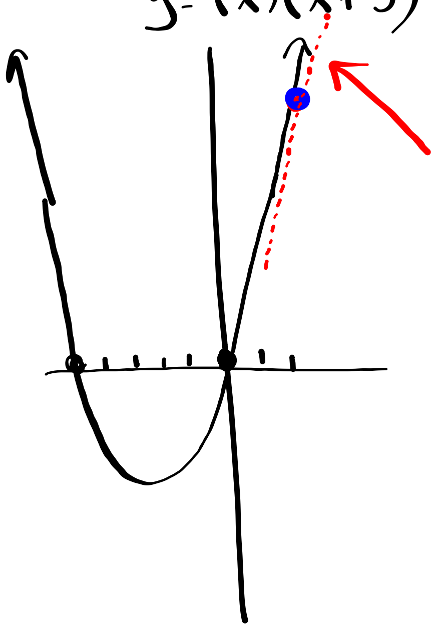
$y = 14$

(iv) Point:

$(2, 14)$

$y = x^2 + 5x$

$y = (x)(x+5)$



The slope of that tangent is 9

Synthetic Sub:

$-27 + 45 + 6 - 24$

$$x^3 + 5x^2 - 2x - 24$$

Find an x value that makes the polynomial equal 0

$(2)^3 + 5(2)^2 - 2(2) - 24$

$8 + 20 - 4 - 24$

0

2	1	5	-2	-24
		2	14	24
x=2	1	7	12	0

$(x-2)(x^2+7x+12)$ ← simple trinomial

$\underline{3} + \underline{4} = 7$

$\underline{3} \times \underline{4} = 12$

$(x-2)(x+3)(x+4)$

Using Factor Theorem

$$x^3 + 5x^2 - 2x - 24$$

$x = -3$

$(-3)^3 + 5(-3)^2 - 2(-3) - 24$

$\boxed{x+3 \text{ is a factor}}$

$-27 + 45 + 6 - 24$

0

$$\begin{array}{r|l}
 & x^2 + 2x - 8 \\
x+3 & x^3 + 5x^2 - 2x - 24 \\
 & -(x^3 + 3x^2) \\
 & \quad 2x^2 - 2x - 24 \\
 & \quad -(2x^2 + 6x) \\
 & \qquad -8x - 24 \\
 & \qquad -(-8x - 24) \\
 & \qquad\qquad 0
\end{array}$$

$(x+3)(x^2+2x-8)$

$(x+3)(x+4)(x-2)$

Curve Sketching:

$$f(x) = \frac{x^2+5x+6}{x^2-9} = \frac{(x+2)\cancel{(x+3)}}{(x-3)\cancel{(x+3)}} = \frac{x+2}{x-3}$$

x int:

$x+2=0$

$x=-2$

$(-2,0)$

y int:

$f(0) = \frac{0+2}{0-3}$

$f(0) = -\frac{2}{3}$

$(0,-2/3)$

VA:

$x-3=0$

$x=3$

HA: $\lim_{x\to\infty} \frac{x+2}{x-3} = 1$

$y=1$

Hole:

$x+3=0$

$x=-3$

$f(-3) = \frac{-3+2}{-3-3}$

$f(-3) = \frac{1}{6}$