Making a Complete Sketch

Example:

Examine the function $f(x) = 3x^5 - 5x^3$ with respect to…

- Intercepts $f(x)$
- ~~Symmetry~~
- Asymptotes (No asymptotes for polynomial functions)
- Intervals of Increase or Decrease $f'(x)$
- Local Maximum and Minimum values $f(x)$
- $f''(x)$ ~~Concavity~~ and ~~Points of Inflection~~ $f(x)$
- Sketch the Curve

$f(x) = 3x^5 - 5x^3$ $\quad f'(x) = 15x^4 - 15x^2$ $\quad f''(x) = 60x^3 - 30x$

$f(x) = x^3(3x^2-5)$ $\quad f'(x) = 15x^2(x^2-1)$ $\quad f''(x) = 30x(2x^2-1)$

$f'(x) = 15x^2(x-1)(x+1)$

① x-int (y=0)

$f(x) = x^3(3x^2-5)$

$0 = x^3(3x^2-5)$

$x^3 = 0$, $x = 0$, (0,0) | $3x^2 - 5 = 0$, $\frac{3x^2}{3} = \frac{5}{3}$, $x^2 = \frac{5}{3}$, $x = \pm\sqrt{5/3}$, (1.29,0) + (-1.29,0)

② y-int (x=0)

$f(x) = 3x^5 - 5x^3$

$f(0) = 3(0)^5 - 5(0)^3$

$f(0) = 0$

(0,0)

③ Intervals of Inc/Dec.

$f'(x) = 15x^2(x-1)(x+1)$

$0 = 15x^2(x-1)(x+1)$

$15x^2 = 0$, $x^2 = 0$, $x = 0$ | $x-1=0$, $x=1$ | $x+1=0$, $x=-1$

CV: x = -1, 0, 1

max min

+ − − +

(-2) -1 (-½) 0 (½) 1 (2)

Increasing on $(-\infty,-1)+(1,\infty)$

Decreasing on $(-1,0)+(0,1)$ or $(-1,1)$

④ max @ x = -1

$f(x) = 3x^5 - 5x^3$

$f(-1) = 3(-1)^5 - 5(-1)^3$

$f(-1) = -3 + 5$

$f(-1) = 2$

(-1,2)

⑤ min @ x = 1

$f(x) = 3x^5 - 5x^3$

$f(1) = 3(1)^5 - 5(1)^3$

$f(1) = 3 - 5$

$f(1) = -2$

(1,-2)

⑥ Intervals of Concavity:

$f''(x) = 30x(2x^2-1)$

$0 = 30x(2x^2-1)$

$30x = 0$, $x = 0$ | $2x^2 - 1 = 0$, $2x^2 = 1$, $x^2 = \frac{1}{2}$, $x = \pm\sqrt{1/2}$, $x = \pm 0.707$

CV: x = -0.7, 0, 0.7

I.P. I.P. I.P.

− + − +

(-1) -0.7 (-0.5) 0 (0.5) 0.7 (1)

CD on $(-\infty,-0.7)+(0,0.7)$

CU on $(-0.7,0)+(0.7,\infty)$

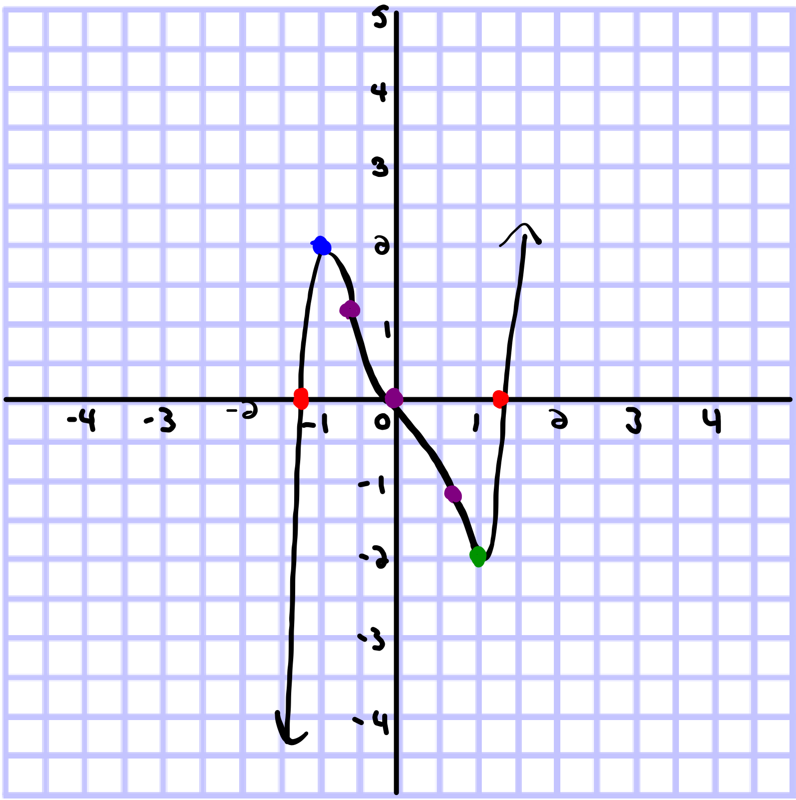
⑦ Inflection Points

$f(x) = 3x^5 - 5x^3$

$f(-0.7) = 3(-0.7)^5 - 5(-0.7)^3 = -0.504 + 1.715 = 1.2$ $\quad (-0.7, 1.2)$

$f(0) = 3(0)^5 - 5(0)^3 = 0 - 0 = 0$ $\quad (0,0)$

$f(0.7) = 3(0.7)^5 - 5(0.7)^3 = 0.504 - 1.715 = -1.2$ $\quad (0.7, -1.2)$


5
4
3
2
1
-4
-3
-2
-1
0
1
2
3
4
-1
-2
-3
-4

Assignment:

$f(x) = x^2 + x^3$ $\quad f'(x) = 2x + 3x^2$ $\quad f''(x) = 2 + 6x$

$f(x) = x^2(1+x)$ $\quad f'(x) = x(2+3x)$ $\quad f''(x) = 2(1+3x)$

① x-int $(y=0)$

$f(x) = x^2(1+x)$

$0 = x^2(1+x)$

$x^2 = 0$	$1+x=0$
$x = 0$	$x = -1$
$(0,0)$	$(-1,0)$

② y-int $(x=0)$

$f(x) = x^2 + x^3$

$f(0) = (0)^2 + (0)^3$

$f(0) = 0 + 0$

$f(0) = 0$

$(0,0)$

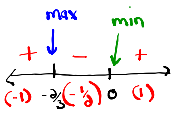
③ Intervals of Inc/Dec.

$f'(x) = x(2+3x)$

$0 = x(2+3x)$

$x = 0$	$2+3x=0$
	$3x = -2$
	$x = -2/3$

CV: $x = -\frac{2}{3}, 0$



Increasing on $(-\infty, -2/3)$ + $(0, \infty)$

Decreasing on $(-2/3, 0)$

④ max @ $x = -\frac{2}{3}$

$f(x) = x^2 + x^3$

$f(-\frac{2}{3}) = (-\frac{2}{3})^2 + (-\frac{2}{3})^3$

$f(-\frac{2}{3}) = \frac{4}{9} - \frac{8}{27}$

$f(-\frac{2}{3}) = \frac{12}{27} - \frac{8}{27} = \frac{4}{27}$

$(-\frac{2}{3}, \frac{4}{27})$ or $(-0.\bar{6}, 0.15)$

⑤ min @ $x = 0$

$f(x) = x^2 + x^3$

$f(0) = (0)^2 + (0)^3$

$f(0) = 0 + 0 = 0$

$(0,0)$

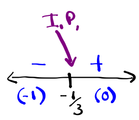
⑥ Intervals of Concavity:

$f''(x) = 2(1+3x)$

$0 = 2(1+3x)$

$2 \neq 0$	$1+3x=0$
	$3x = -1$
	$x = -\frac{1}{3}$

CV: $x = -\frac{1}{3}$



CD on $(-\infty, -\frac{1}{3})$

CU on $(-\frac{1}{3}, \infty)$

⑦ Inflection Point: @ $x = -\frac{1}{3}$

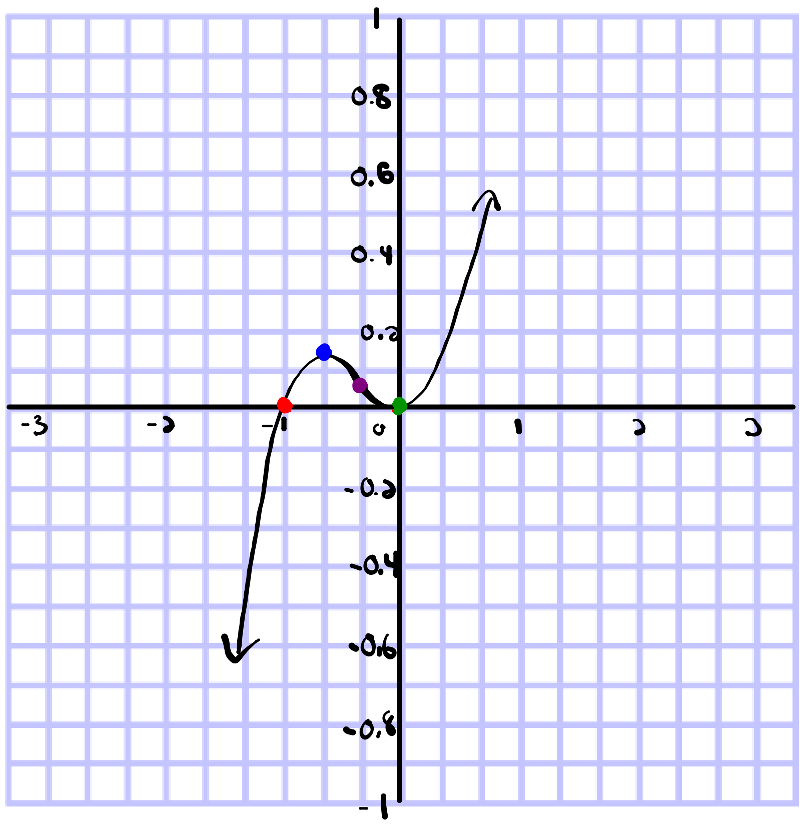
$f(x) = x^2 + x^3$

$f(-\frac{1}{3}) = (-\frac{1}{3})^2 + (-\frac{1}{3})^3$

$f(-\frac{1}{3}) = \frac{1}{9} - \frac{1}{27}$

$f(-\frac{1}{3}) = \frac{3}{27} - \frac{1}{27} = \frac{2}{27}$

$(-\frac{1}{3}, \frac{2}{27})$ or $(-0.\bar{3}, 0.07)$



1
0.8
0.6
0.4
0.2
-3
-2
-1
0
1
2
3
-0.2
-0.4
-0.6
-0.8
-1

homework

Examine the function $f(x) = \frac{x^2}{1-x^2}$ with respect to... $f'(x) = \frac{2x}{(1-x^2)^2}$

- Intercepts $f(x)$
- ~~Symmetry~~
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

$f''(x) = \frac{2(1+3x^2)}{(1-x^2)^3}$

① x-int (y=0)

$f(x) = \frac{x^2}{1-x^2}$

$0 = \frac{x^2}{1-x^2}$

$0 = x^2$

$0 = x$

(0,0)

② y-int (x=0)

$f(x) = \frac{x^2}{1-x^2}$

$f(0) = \frac{(0)^2}{1-(0)^2}$

$f(0) = \frac{0}{1} = 0$

(0,0)

③ Vertical Asymptote:
(zeroes of the denominator)

$f(x) = \frac{x^2}{1-x^2}$

VA: $1-x^2=0$

$(1-x)(1+x)=0$

$1-x=0$ | $1+x=0$

$\boxed{1=x}$ | $\boxed{x=-1}$

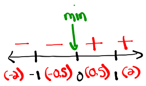
④ Horizontal Asymptote:

$\lim_{x\to\infty} \frac{x^2}{1-x^2} = \frac{1}{-1} = -1$

$\boxed{y=-1}$

⑤ Intervals of Inc/Dec.

$f'(x) = \frac{2x}{(1-x^2)^2}$



$2x=0$ | $(1-x^2)^2=0$

$x=0$ | $1-x^2=0$

| $1=x^2$

| $\pm1=x$

CV: $x=-1,0,1$

Increasing on $(0,\infty)$

Decreasing on $(-\infty,0)$

⑥ min @ x=0

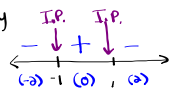
$f(x) = \frac{x^2}{1-x^2}$

$f(0) = \frac{(0)^2}{1-(0)^2} = 0$

(0,0)

⑦ Intervals of concavity

$f''(x) = \frac{2(1+3x^2)}{(1-x^2)^3}$



$2(1+3x^2)=0$ | $(1-x^2)^3=0$

$1+3x^2=0$ | $1-x^2=0$

$3x^2=-1$ | $1=x^2$

$x^2=-\frac{1}{3}$ | $\pm1=x$

Not Possible
(Numerator is always positive)

CD on $(-\infty,-1)+(1,\infty)$

CU on $(-1,1)$

⑧ Inflection Points:

when x=-1

$f(x) = \frac{x^2}{1-x^2}$

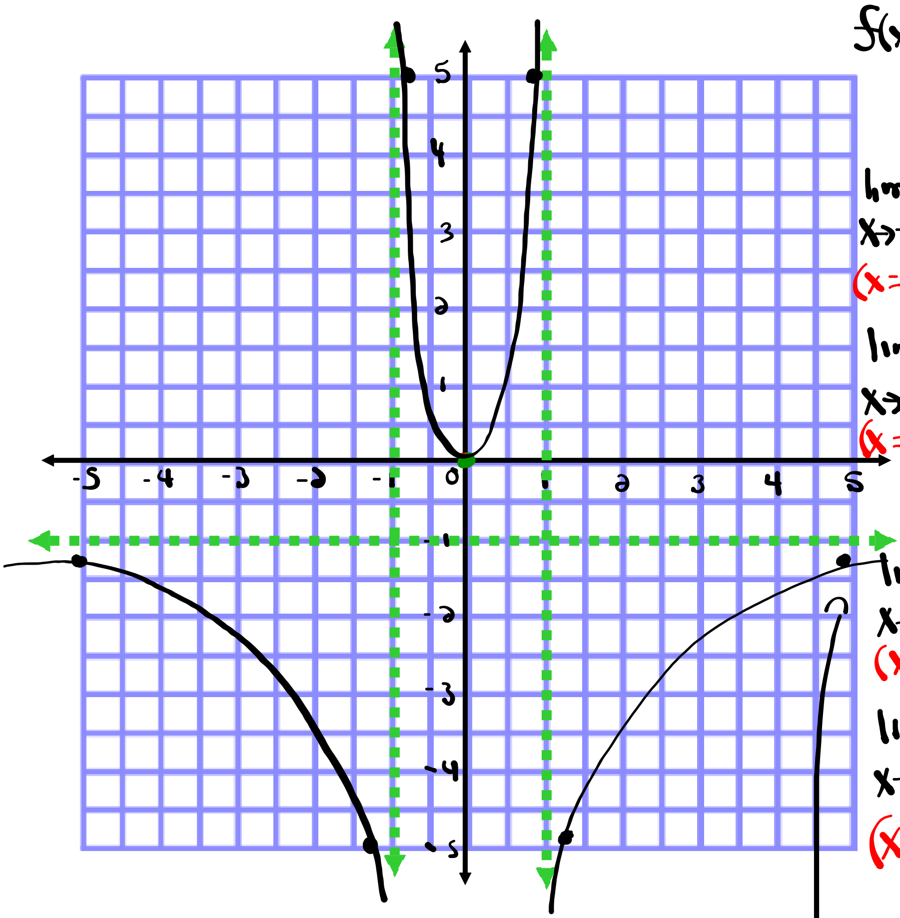
$f(-1) = \frac{(-1)^2}{1-(-1)^2} = \frac{1}{0} = $ und.

when x=1

$f(x) = \frac{x^2}{1-x^2}$

$f(1) = \frac{(1)^2}{1-(1)^2} = \frac{1}{0} = $ und

* No Inflection Points x=±1 are the Vertical Asymptotes.



$f(x) = \dfrac{x^2}{(1-x)(1+x)}$

$\lim\limits_{x \to -1^-} f(x) = \dfrac{(+)}{(+)(-)} = -\infty$

(x=-1.1)

$\lim\limits_{x \to -1^+} f(x) = \dfrac{(+)}{(+)(+)} = +\infty$

(x=-0.9)

$\lim\limits_{x \to 1^-} f(x) = \dfrac{(+)}{(+)(+)} = +\infty$

(x=0.9)

$\lim\limits_{x \to 1^+} f(x) = \dfrac{(+)}{(-)(+)} = -\infty$

(x=1.1)

$f(5) = \dfrac{(5)^2}{1-(5)^2} = \dfrac{25}{-24}$

$$f(x) = \frac{x^2}{1-x^2}$$

$$f'(x) = \frac{2x(1-x^2) + 2x(x^2)}{(1-x^2)^2}$$

$$f'(x) = \frac{2x - 2x^3 + 2x^3}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

$$f''(x) = \frac{2(1-x^2)^2 - 2x(2)(1-x^2)(-2x)}{(1-x^2)^4}$$

$$f''(x) = \frac{2(1-x^2)^2 + 8x^2(1-x^2)}{(1-x^2)^4}$$

$$f''(x) = \frac{2(1-x^2)\left[(1-x^2)+4x^2\right]}{(1-x^2)^4} =$$

$$f''(x) = \frac{2\cancel{(1-x^2)}(1+3x^2)}{(1-x^2)^{\cancel{4}\,3}} = \frac{2(1+3x^2)}{(1-x^2)^3}$$

homework

Examine the function $f(x) = x^4 - 4x^3$ with respect to…

- Intercepts
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

Solution

homework

Examine the function $f(x) = \frac{x^2}{x-7}$ with respect to…

- Intercepts
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

① x-int (y=0)

$f(x) = \frac{x^2}{x-7}$

$(x-7)\ 0 = \frac{x^2}{\cancel{x-7}}\ \cancel{(x-7)}$

$0 = x^2$

$0 = x$

$(0,0)$

② y-int (x=0)

$f(x) = \frac{x^2}{x-7}$

$f(0) = \frac{(0)^2}{(0)-7} = \frac{0}{-7} = 0$

$y = 0$

$(0,0)$

③ Symmetry:

$f(x) = \frac{x^2}{x-7}$

$f(-x) = \frac{(-x)^2}{(-x)-7}$

$f(-x) = \frac{x^2}{-x-7}$

No symmetry

④ VA: (denom = 0)

$x-7=0$

$x=7$

$\lim_{x\to7^-} \frac{x^2}{x-7} = \frac{(+)}{(-)} = -\infty$ (6.9)

$\lim_{x\to7^+} \frac{x^2}{x-7} = \frac{(+)}{(+)} = +\infty$ (x=7.1)

⑤ SA:

$$\begin{array}{r} x+7 \\ x-7\,\overline{)\,x^2\qquad} \\ -(x^2-7x) \\ \hline 7x \\ -(7x-49) \\ \hline 49\ R \end{array}$$

$y = x+7$

$m = \frac{1}{1}\ \frac{rise}{run}$

$b = 7$ y int

⑥ Intervals of Inc/Dec:

$f'(x) = \frac{x(x-14)}{(x-7)^2}$

max, neither, min

+ | − | − | +

(−1) 0 (1) 7 (8) 14 (15)

CV: $x=0$ | $x-14=0$, $x=14$ | $(x-7)^2=0$, $x-7=0$, $x=7$

Increasing on $(-\infty,0)+(14,\infty)$

$x<0$ + $x>14$

Decreasing on $(0,14)$

$0<x<14$

⑦ Local max/min

$f(x) = \frac{x^2}{x-7}$

when $x=0$

$f(0) = \frac{(0)^2}{(0)-7} = \frac{0}{-7} = 0$

$(0,0)$

local max @ $(0,0)$

When $x=14$

$f(14) = \frac{(14)^2}{(14)-7} = \frac{196}{7} = 28$

$(14,28)$

local min @ $(14,28)$

⑧ Intervals of Concavity:

$f''(x) = \frac{98}{(x-7)^3}$

I.P.

− | +

(6) (7) (8)

CV: $98 \neq 0$ | $(x-7)^3=0$, $x-7=0$, $x=7$

Concave down on $(-\infty,7)$

$x<7$

Concave up on $(7,\infty)$

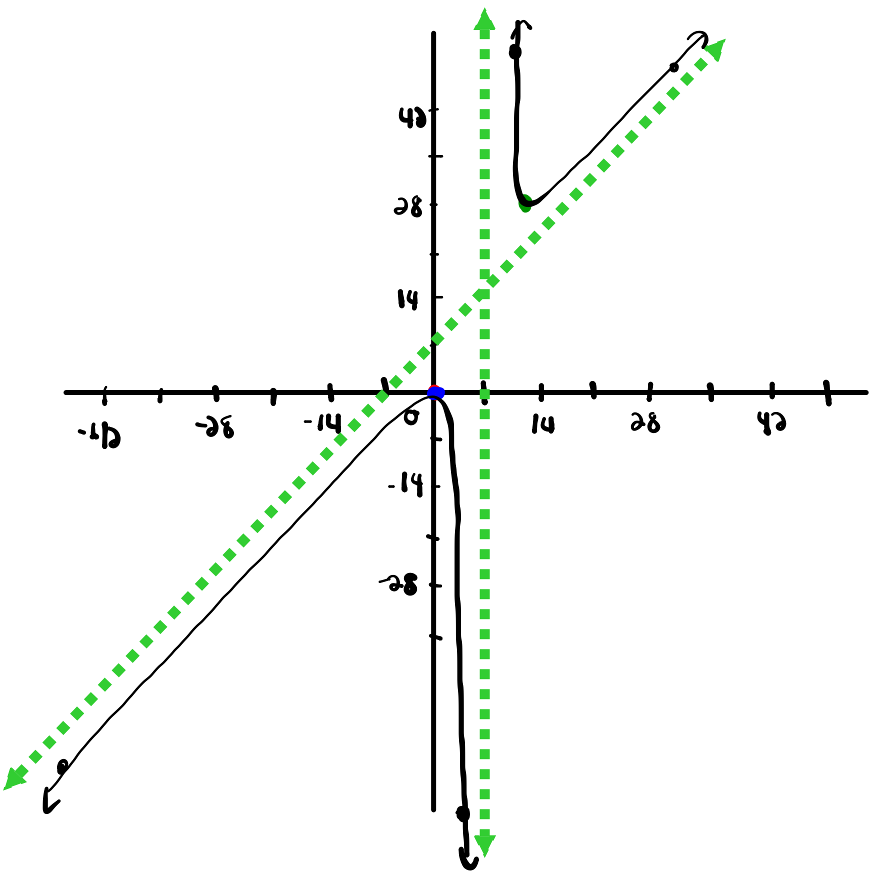
$x>7$

⑨ I.P. $(x=7)$

$f(x) = \frac{x^2}{x-7}$

$f(7) = \frac{(7)^2}{(7)-7} = \frac{49}{0} = DNE$

$x=7$ is the vertical asymptote


42
28
14
-14
-28
-42
-28
-14
0
14
28
42