

Making a Complete Sketch

- ① Plot all points: x -int, y -int, max, min, I.P.,
behaviour near VA.
- ② Plot all asymptotes
- ③ use intervals of inc/dec and concavity to connect everything

Example:

Examine the function $f(x) = 3x^5 - 5x^3$ with respect to...

- Intercepts $f(x)$
- ~~Symmetry~~
- Asymptotes (No asymptotes for polynomial functions)
- Intervals of Increase or Decrease $f'(x)$
- Local Maximum and Minimum values $f(x)$
- ~~Concavity and Points of Inflection $f''(x)$~~
- Sketch the Curve

$$f(x) = 3x^5 - 5x^3 \quad f'(x) = 15x^4 - 15x^2 \quad f''(x) = 60x^3 - 30x$$

$$f(x) = x^3(3x^2 - 5) \quad f'(x) = 15x^2(x^2 - 1) \quad f''(x) = 30x(x^2 - 1)$$

$$f'(x) = 15x^2(x-1)(x+1)$$

① x-int ($y=0$)

$$f(x) = x^3(3x^2 - 5)$$

$$0 = x^3(3x^2 - 5)$$

$$x^3 = 0 \quad 3x^2 - 5 = 0$$

$$x = 0 \quad 3x^2 = 5$$

$$(0,0) \quad x^2 = \frac{5}{3}$$

$$x = \pm\sqrt{\frac{5}{3}}$$

$$(1.29, 0) \quad + (-1.29, 0)$$

② y-int ($x=0$)

$$f(x) = 3x^5 - 5x^3$$

$$f(0) = 3(0)^5 - 5(0)^3$$

$$f(0) = 0$$

$$(0,0)$$

③ Intervals of Inc/Dec.

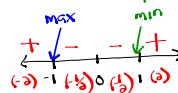
$$f'(x) = 15x^2(x-1)(x+1)$$

$$0 = 15x^2(x-1)(x+1)$$

$$15x^2 = 0 \quad x-1 = 0 \quad x+1 = 0$$

$$x^2 = 0 \quad x = 1 \quad x = -1$$

$$x = 0$$



Increasing on $(-\infty, -1) + (1, \infty)$
 Decreasing on $(-1, 0) + (0, 1)$
 or $(-1, 1)$

CV: $x = -1, 0, 1$

④ max @ $x = -1$

$$f(x) = 3x^5 - 5x^3$$

$$f(-1) = 3(-1)^5 - 5(-1)^3$$

$$f(-1) = -3 + 5$$

$$f(-1) = 2$$

$$(-1, 2)$$

⑤ min @ $x = 1$

$$f(x) = 3x^5 - 5x^3$$

$$f(1) = 3(1)^5 - 5(1)^3$$

$$f(1) = 3 - 5$$

$$f(1) = -2$$

$$(1, -2)$$

⑥ Intervals of Concavity:

$$f''(x) = 30x(x^2 - 1)$$

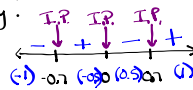
$$0 = 30x(x^2 - 1)$$

$$30x = 0 \quad x^2 - 1 = 0$$

$$x = 0 \quad x^2 = 1$$

$$x = \pm\sqrt{1}$$

$$x = \pm 1$$



CD on $(-\infty, -0.7) + (0, 0.7)$
 CU on $(-0.7, 0) + (0.7, \infty)$

CV: $x = -0.7, 0, 0.7$

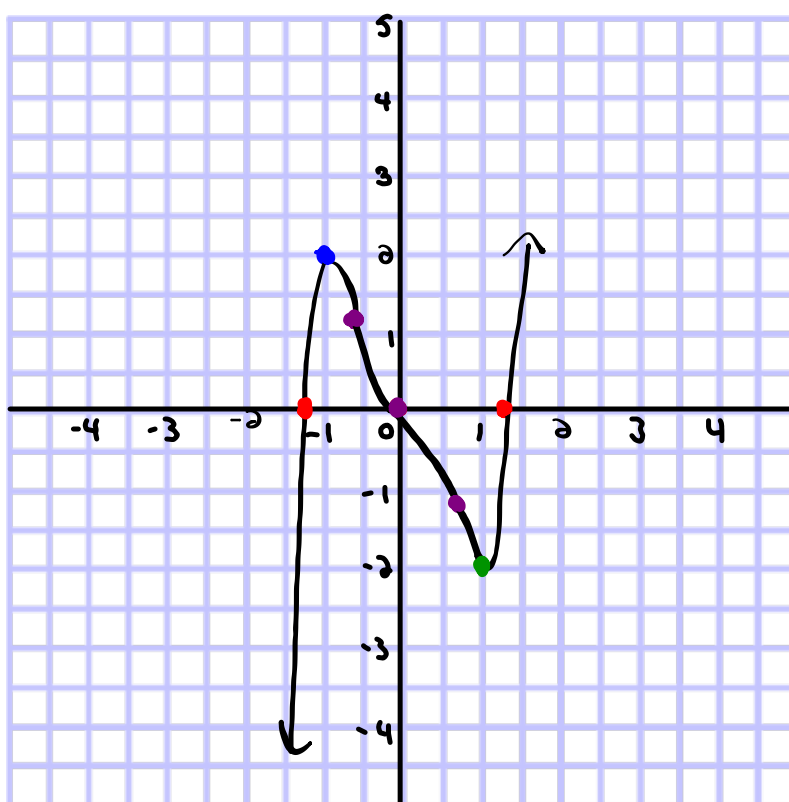
⑦ Inflection Points

$$f(x) = 3x^5 - 5x^3$$

$$f(-0.7) = 3(-0.7)^5 - 5(-0.7)^3 = -0.504 + 1.715 = 1.2 \quad (-0.7, 1.2)$$

$$f(0) = 3(0)^5 - 5(0)^3 = 0 - 0 = 0 \quad (0, 0)$$

$$f(0.7) = 3(0.7)^5 - 5(0.7)^3 = 0.504 - 1.715 = -1.2 \quad (0.7, -1.2)$$



Assignment:

$$f(x) = x^2 + x^3 \quad f'(x) = 2x + 3x^2 \quad f''(x) = 2 + 6x$$

$$f(x) = x^2(1+x) \quad f'(x) = x(2+3x) \quad f''(x) = 2(1+3x)$$

① x-int ($y=0$) ② y-int ($x=0$)

$$f(x) = x^2(1+x)$$

$$0 = x^2(1+x)$$

$$\begin{array}{l|l} x^2=0 & 1+x=0 \\ x=0 & x=-1 \\ (0,0) & (-1,0) \end{array}$$

$$f(x) = x^2 + x^3$$

$$f(0) = (0)^2 + (0)^3$$

$$f(0) = 0 + 0$$

$$f(0) = 0$$

$$(0,0)$$

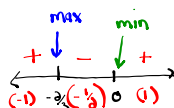
③ Intervals of Inc/Dec.

$$f'(x) = x(2+3x)$$

$$0 = x(2+3x)$$

$$\begin{array}{l|l} x=0 & 2+3x=0 \\ & 3x=-2 \\ & x=-\frac{2}{3} \end{array}$$

$$\text{CV: } x = -\frac{2}{3}, 0$$

Increasing on $(-\infty, -\frac{2}{3}) \cup (0, \infty)$ Decreasing on $(-\frac{2}{3}, 0)$ ④ max @ $x = -\frac{2}{3}$

$$f(x) = x^2 + x^3$$

$$f\left(-\frac{2}{3}\right) = \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^3$$

$$f\left(-\frac{2}{3}\right) = \frac{4}{9} - \frac{8}{27}$$

$$f\left(-\frac{2}{3}\right) = \frac{12}{27} - \frac{8}{27} = \frac{4}{27}$$

$$\left(-\frac{2}{3}, \frac{4}{27}\right) \text{ or } (-0.6, 0.15)$$

⑤ min @ $x = 0$

$$f(x) = x^2 + x^3$$

$$f(0) = (0)^2 + (0)^3$$

$$f(0) = 0 + 0 = 0$$

$$(0,0)$$

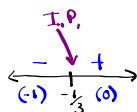
⑥ Intervals of Concavity:

$$f''(x) = 2(1+3x)$$

$$0 = 2(1+3x)$$

$$\begin{array}{l|l} 2 \neq 0 & 1+3x=0 \\ & 3x=-1 \\ & x=-\frac{1}{3} \end{array}$$

$$\text{CV: } x = -\frac{1}{3}$$

CD on $(-\infty, -\frac{1}{3})$ CU on $(-\frac{1}{3}, \infty)$ ⑦ Inflection Point: @ $x = -\frac{1}{3}$

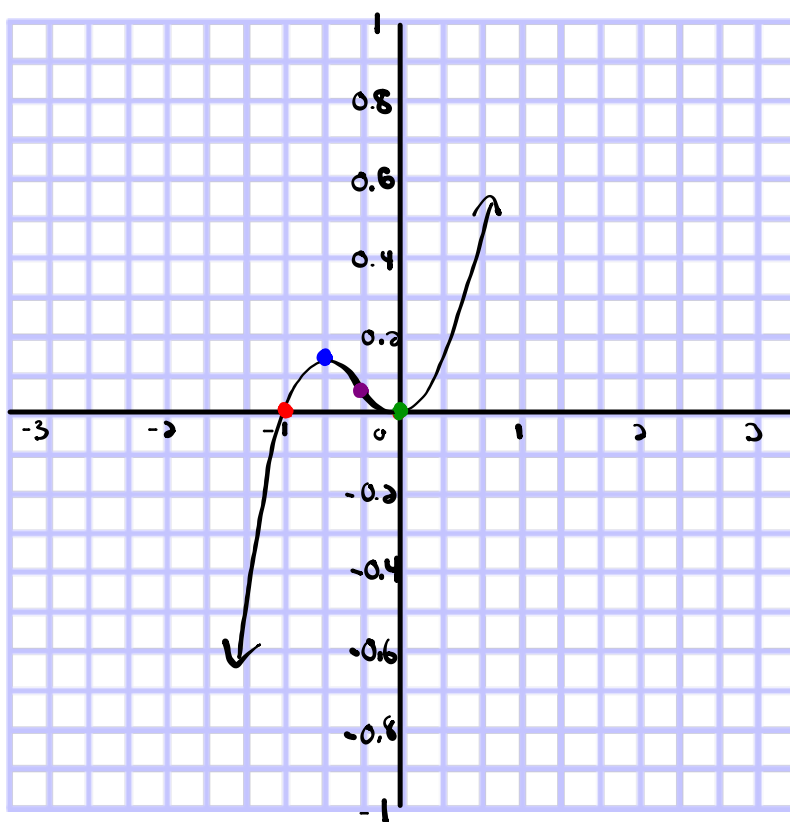
$$f(x) = x^2 + x^3$$

$$f\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^3$$

$$f\left(-\frac{1}{3}\right) = \frac{1}{9} - \frac{1}{27}$$

$$f\left(-\frac{1}{3}\right) = \frac{3}{27} - \frac{1}{27} = \frac{2}{27}$$

$$\left(-\frac{1}{3}, \frac{2}{27}\right) \text{ or } (-0.3, 0.07)$$



homework

Examine the function $f(x) = \frac{x^2}{1-x^2}$ with respect to... $f'(x) = \frac{2x}{(1-x^2)^2}$

- Intercepts $f(x)$
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

$$f''(x) = \frac{2(1+3x^2)}{(1-x^2)^3}$$

Ⓐ x-int ($y=0$)

$$f(x) = \frac{x^2}{1-x^2}$$

$$0 = \frac{x^2}{1-x^2}$$

$$0 = x^2$$

$$0 = x$$

$$(0,0)$$

Ⓑ y-int ($x=0$)

$$f(x) = \frac{x^2}{1-x^2}$$

$$f(0) = \frac{0^2}{1-0^2}$$

$$f(0) = \frac{0}{1} = 0$$

$$(0,0)$$

Ⓒ Vertical Asymptote: (zeros of the denominator)

$$f(x) = \frac{x^2}{1-x^2}$$

$$\text{VA: } 1-x^2=0$$

$$(1-x)(1+x)=0$$

$$1-x=0 \quad 1+x=0$$

$$1=x \quad x=-1$$

Ⓓ Horizontal Asymptote:

$$\lim_{x \rightarrow \infty} \frac{x^2}{1-x^2} = \frac{1}{-1} = -1$$

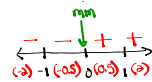
$$y = -1$$

Ⓔ Intervals of Inc/Dec.

$$f'(x) = \frac{2x}{(1-x^2)^2}$$

$$\begin{array}{l} 2x=0 \\ x=0 \end{array} \left| \begin{array}{l} (1-x^2)^2=0 \\ 1-x^2=0 \\ 1-x^2 \\ \pm 1=x \end{array} \right.$$

$$\text{Cr: } x = -1, 0, 1$$



Increasing on $(0, \infty)$
Decreasing on $(-\infty, 0)$

Ⓕ min @ $x=0$

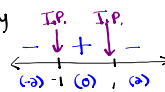
$$f(x) = \frac{x^2}{1-x^2}$$

$$f(0) = \frac{0^2}{1-0^2} = 0$$

$$(0,0)$$

Ⓖ Intervals of concavity

$$f''(x) = \frac{2(1+3x^2)}{(1-x^2)^3}$$



$$\begin{array}{l} 2(1+3x^2)=0 \\ 1+3x^2=0 \\ 3x^2=-1 \\ x^2=-\frac{1}{3} \end{array} \left| \begin{array}{l} (1-x^2)^3=0 \\ 1-x^2=0 \\ 1-x^2 \\ \pm 1=x \end{array} \right.$$

Not Possible

(Numerator is always positive)

CO on $(-\infty, -1) \cup (1, \infty)$
CU on $(-1, 1)$

Ⓖ Inflection Points:

when $x=-1$

$$f(x) = \frac{x^2}{1-x^2}$$

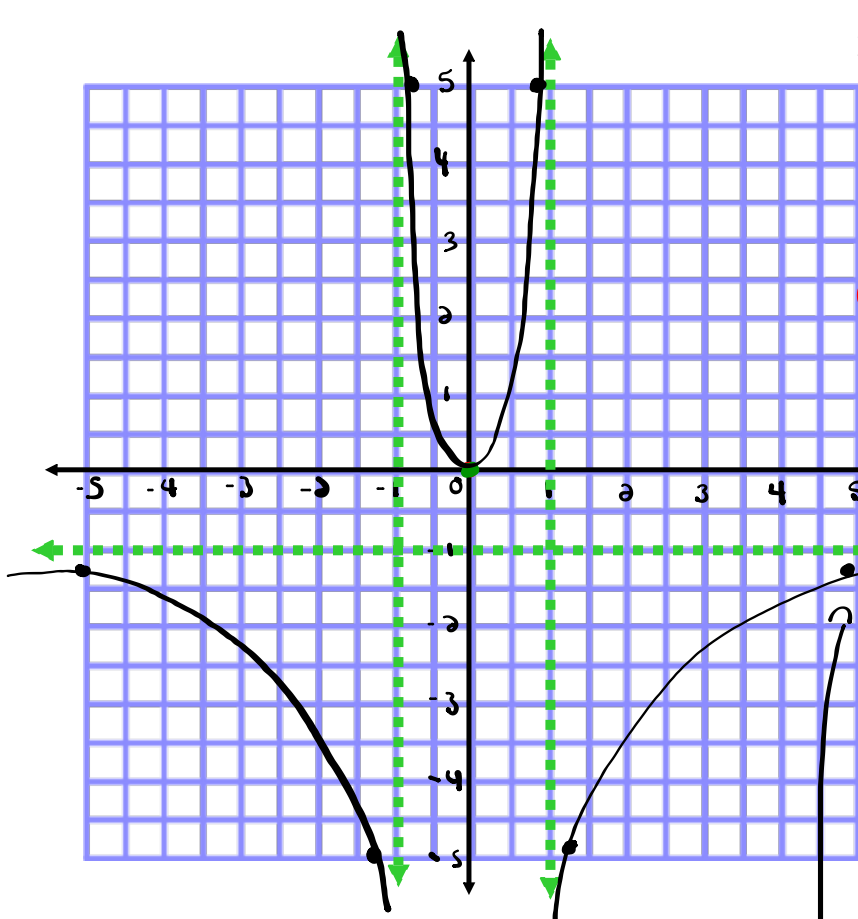
$$f(-1) = \frac{(-1)^2}{1-(-1)^2} = \frac{1}{0} = \text{und.}$$

when $x=1$

$$f(x) = \frac{x^2}{1-x^2}$$

$$f(1) = \frac{(1)^2}{1-(1)^2} = \frac{1}{0} = \text{und.}$$

* No Inflection Points $x = \pm 1$ are the Vertical Asymptotes.



$$f(x) = \frac{x^2}{(1-x)(1+x)}$$

$$\lim_{x \rightarrow 1^-} f(x) = \frac{(+)}{(+)(-)} = -\infty$$

(x = -1.1)

$$\lim_{x \rightarrow 1^+} f(x) = \frac{(+)}{(+)(+)} = +\infty$$

(x = -0.9)

$$\lim_{x \rightarrow 1^-} f(x) = \frac{(+)}{(+)(+)} = +\infty$$

(x = 0.9)

$$\lim_{x \rightarrow 1^+} f(x) = \frac{(+)}{(-)(+)} = -\infty$$

(x = 1.1)

$$f(5) = \frac{(5)^2}{1-(5)^2} = \frac{25}{-24}$$

$$f(x) = \frac{x^2}{1-x^2}$$

$$f'(x) = \frac{2x(1-x^2) + 2x(x^2)}{(1-x^2)^2}$$

$$f'(x) = \frac{2x - 2x^3 + 2x^3}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

$$f''(x) = \frac{2(1-x^2)^2 - 2x(2)(1-x^2)(-2x)}{(1-x^2)^4}$$

$$f''(x) = \frac{2(1-x^2)^2 + 8x^2(1-x^2)}{(1-x^2)^4}$$

$$f''(x) = \frac{2(1-x^2)[(1-x^2) + 4x^2]}{(1-x^2)^4} =$$

$$f''(x) = \frac{2\cancel{(1-x^2)}(1+3x^2)}{(1-x^2)\cancel{4}_3} = \frac{2(1+3x^2)}{(1-x^2)^3}$$

Review:

① $f(x) = \frac{8(x-2)}{x^2}$ $f'(x) = \frac{-8(x-4)}{x^3}$ $f''(x) = \frac{16(x-6)}{x^4}$

① Intercepts:

x-int (y=0)
 $\frac{0}{1} = \frac{8(x-2)}{x^2}$
 $8(x-2) = 0$
 $x-2 = 0$
 $x = 2$
(2,0)

y-int (x=0)
 $f(0) = \frac{8(0-2)}{(0)^2}$
 $f(0) = \frac{-16}{0}$
 $f(0) = \text{undefined}$
 No y-int

② Symmetry:

$f(x) = \frac{8(x-2)}{x^2} = \frac{8x-16}{x^2}$

$f(-x) = \frac{8(-x-2)}{(-x)^2}$

$f(-x) = \frac{8(-x-2)}{x^2}$

$f(-x) = \frac{-8(x+2)}{x^2} = \frac{-8x-16}{x^2}$

No symmetry

③ VA: (zeros of denom)

$f(x) = \frac{8(x-2)}{x^2}$
 $x^2 = 0$
x=0

Check behavior of f(x) near VA:

$\lim_{x \rightarrow 0^-} f(x) = \frac{(-)}{(+)} = -\infty$
 (x=0.01)
 $\lim_{x \rightarrow 0^+} f(x) = \frac{(-)}{(+)} = -\infty$
 (x=0.01)

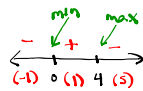
④ HA:

$f(x) = \frac{8(x-2)}{x^2} = \frac{8x-16}{x^2}$

$\lim_{x \rightarrow \infty} \frac{8x-16}{x^2} = 0$

y=0

⑤ Intervals of Inc/Dec



$f'(x) = \frac{-8(x-4)}{x^3}$

Increasing on (0,4)
 Decreasing on $(-\infty, 0) \cup (4, \infty)$

CV:
 $-8(x-4) = 0 \mid x^3 = 0$
 $x-4 = 0 \mid x = 0$
 $x = 4$

⑥ Max/Min:

$f(x) = \frac{8(x-2)}{x^2}$

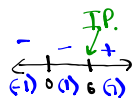
$f(0) = \frac{8(0-2)}{(0)^2} = \text{undefined}$

no min
 x=0 is the VA

$f(4) = \frac{8(4-2)}{(4)^2} = \frac{16}{16} = 1$

max at (4,1)

⑦ Concavity:



$f''(x) = \frac{16(x-6)}{x^4}$

CU on (6,∞)
 CD on $(-\infty, 0) \cup (0, 6)$

CV:
 $16(x-6) = 0 \mid x^4 = 0$
 $x-6 = 0 \mid x = 0$
 $x = 6$

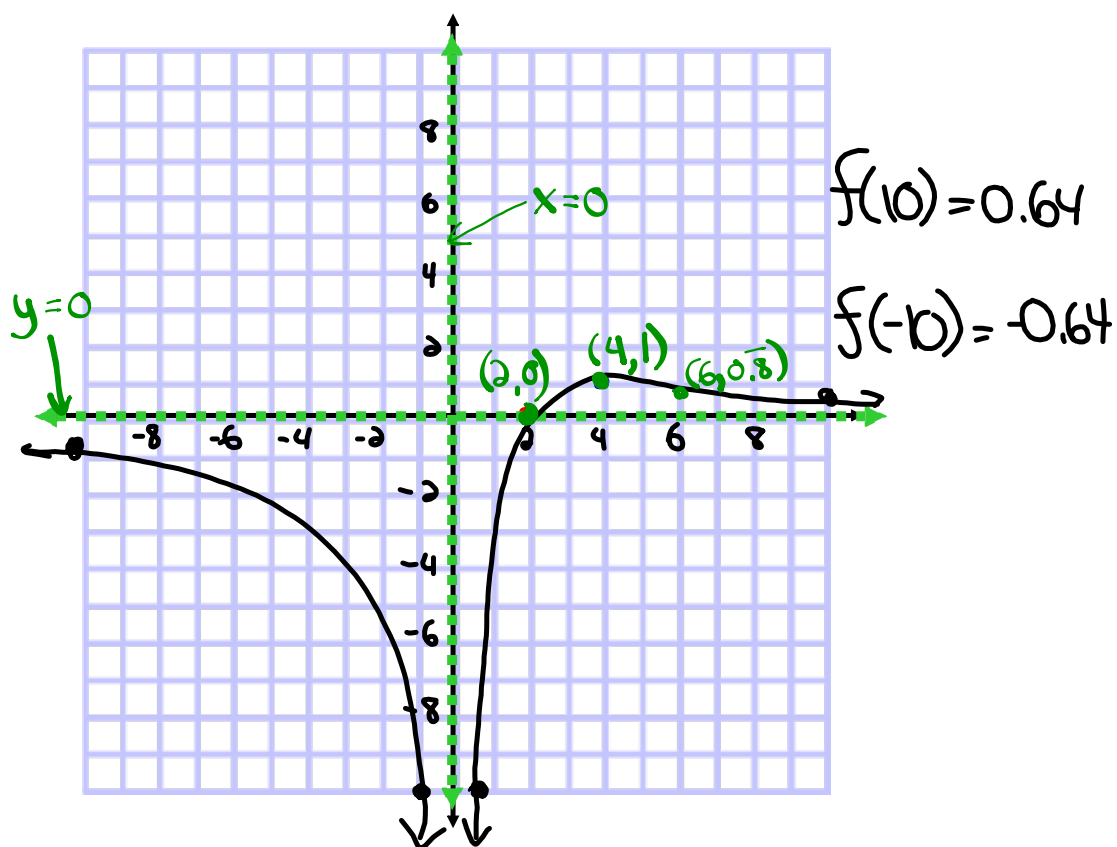
⑧ Inflection Point

$f(x) = \frac{8(x-2)}{x^2}$

$f(6) = \frac{8(6-2)}{6^2} = \frac{32}{36}$

(6, 0.8)

Review:



Review:

Examine the function $f(x) = x^4 - 4x^3$ with respect to...

- Intercepts
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

$$f(x) = x^4 - 4x^3 \quad f'(x) = 4x^3 - 12x^2 \quad f''(x) = 12x^2 - 24x$$

$$f(x) = x^3(x-4) \quad f'(x) = 4x^2(x-3) \quad f''(x) = 12x(x-2)$$

① x-int ($y=0$)

$$f(x) = x^3(x-4)$$

$$0 = x^3(x-4)$$

$$x^3 = 0 \quad | \quad x-4 = 0$$

$$x = 0 \quad | \quad x = 4$$

$$(0,0) \quad | \quad (4,0)$$

② y-int ($x=0$)

$$f(x) = x^4 - 4x^3$$

$$f(0) = (0)^4 - 4(0)^3$$

$$f(0) = 0$$

$$(0,0)$$

③ Intervals of Inc/Dec:

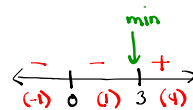
$$f'(x) = 4x^2(x-3)$$

$$4x^2 = 0 \quad | \quad x-3 = 0$$

$$x^2 = 0 \quad | \quad x = 3$$

$$x = 0 \quad | \quad x = 3$$

$$CV: x = 0, 3$$

Decreasing on $(-\infty, 3)$ Increasing on $(3, \infty)$ ④ min @ $x=3$

$$f(x) = x^4 - 4x^3$$

$$f(3) = (3)^4 - 4(3)^3$$

$$f(3) = 81 - 108$$

$$f(3) = -27$$

$$(3, -27)$$

⑤ No local max

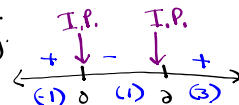
⑥ Intervals of concavity:

$$f''(x) = 12x(x-2)$$

$$12x = 0 \quad | \quad x-2 = 0$$

$$x = 0 \quad | \quad x = 2$$

$$CV: x = 0, 2$$

CO on $(0, 2)$ CU on $(-\infty, 0)$ and $(2, \infty)$

⑦ Inflection points:

when $x=0$

$$f(x) = x^4 - 4x^3$$

$$f(0) = (0)^4 - 4(0)^3$$

$$f(0) = 0$$

$$(0,0)$$

when $x=2$

$$f(x) = x^4 - 4x^3$$

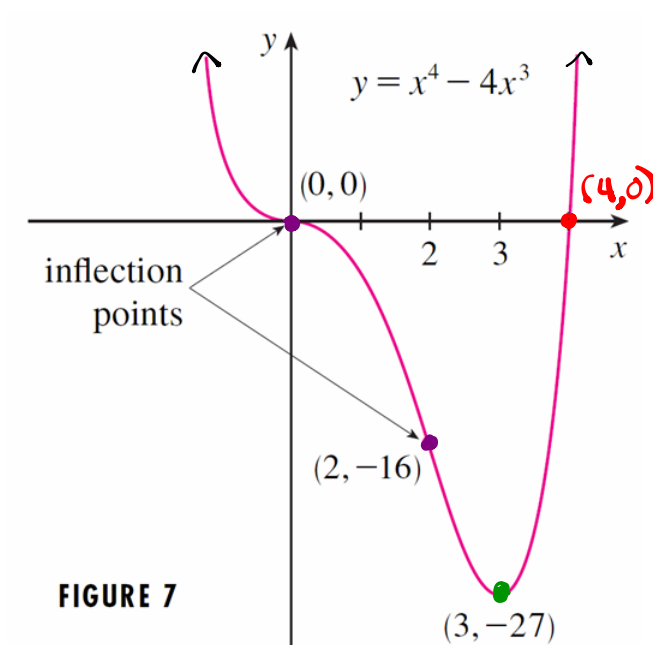
$$f(2) = (2)^4 - 4(2)^3$$

$$f(2) = 16 - 32$$

$$f(2) = -16$$

$$(2, -16)$$

Review:



homework

Examine the function $f(x) = \frac{x^2}{x-7}$ with respect to...

- Intercepts
- Symmetry
- Asymptotes
- Intervals of Increase or Decrease
- Local Maximum and Minimum values
- Concavity and Points of Inflection
- Sketch the Curve

<p>① x-int (y=0)</p> $f(x) = \frac{x^2}{x-7}$ $(x-7) \cdot 0 = \frac{x^2}{x-7} \cdot (x-7)$ $0 = \frac{x^2}{x-7} \cdot (x-7)$ $0 = x^2$ $0 = x$ <p>(0,0)</p>	<p>② y-int (x=0)</p> $f(x) = \frac{x^2}{x-7}$ $f(0) = \frac{0^2}{0-7} = \frac{0}{-7} = 0$ <p>(0,0)</p>	<p>③ Symmetry:</p> $f(x) = \frac{x^2}{x-7}$ $f(-x) = \frac{(-x)^2}{(-x)-7}$ $f(-x) = \frac{x^2}{-x-7}$ <p>No symmetry</p>
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④ VA: (denom=0)

$x-7=0$	$x=7$	$\lim_{x \rightarrow 7^-} \frac{x^2}{x-7} = \frac{49}{(-)} = -\infty$	$\lim_{x \rightarrow 7^+} \frac{x^2}{x-7} = \frac{49}{(+)} = +\infty$
		(6,9)	(6,9)

⑤ SA:

$\frac{x+7}{-(x^2-7x)}$	$y = x+7$	$m = \frac{1}{1}$ rise	$b = 7$ y-int
$-\frac{7x-49}{49}$		run	

⑥ Intervals of Inc/Dec:

$f'(x) = \frac{x(x-14)}{(x-7)^2}$

CV: $x=0$ | $x-14=0$ | $(x-7)^2=0$

$x=14$ | $x-7=0$ | $x=7$

Increasing on $(-\infty, 0) \cup (14, \infty)$
 $x < 0$ + $x > 14$

Decreasing on $(0, 14)$
 $0 < x < 14$

⑦ Local max/min

$f(x) = \frac{x^2}{x-7}$	
When $x=0$	When $x=14$
$f(0) = \frac{0^2}{0-7} = \frac{0}{-7} = 0$	$f(14) = \frac{(14)^2}{(14)-7} = \frac{196}{7} = 28$
(0,0)	(14,28)
local max @ (0,0)	local min @ (14,28)

⑧ Intervals of Concavity:

$f''(x) = \frac{98}{(x-7)^3}$

CV: $98 \neq 0$ | $(x-7)^3=0$

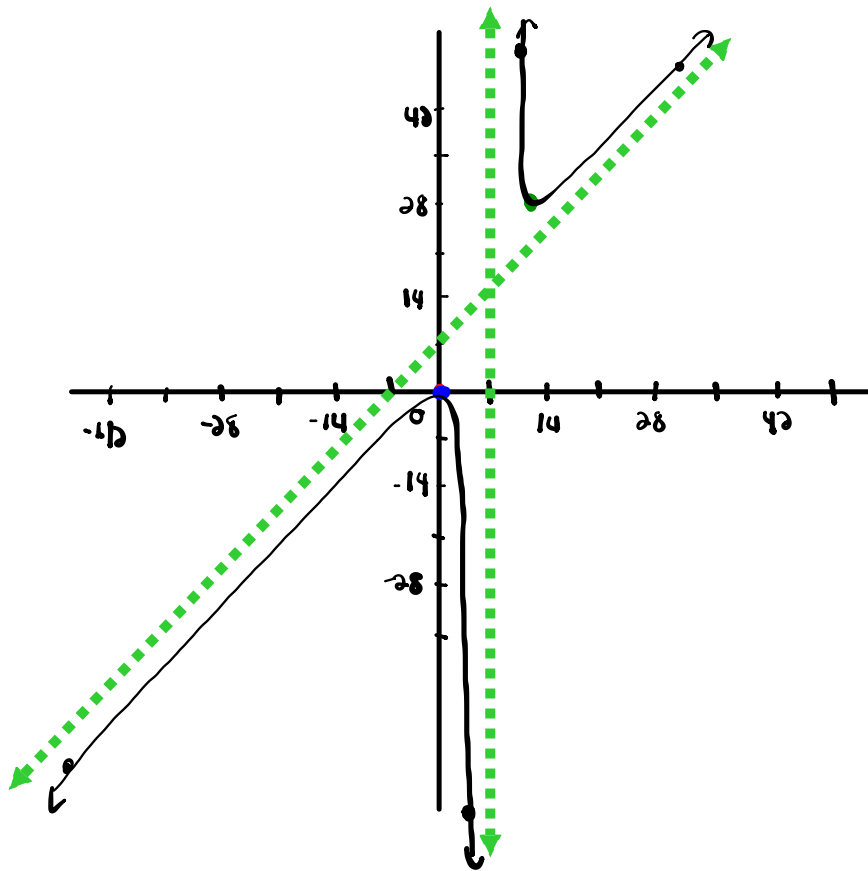
$x-7=0$ | $x=7$

Concave down on $(-\infty, 7)$
 $x < 7$

Concave up on $(7, \infty)$
 $x > 7$

⑨ I.P. (x=7)

$f(x) = \frac{x^2}{x-7}$	$x=7$ is the vertical asymptote.
$f(7) = \frac{7^2}{7-7} = \frac{49}{0} = \text{DNE}$	



Review:

$$f(x) = x^4 - 3x^3 + 3x^2 - x$$

$$(1)^3 - 3(1)^2 + 3(1) - 1$$

$$1 - 3 + 3 - 1$$

① x-int (y=0)

$$0 = x^4 - 3x^3 + 3x^2 - x$$

$$0 = x(x^3 - 3x^2 + 3x - 1) \text{ syn. sub.}$$

$$0 = x(x-1)(x^2 - 2x + 1) \text{ simple tri.}$$

$$\frac{-1}{-1} x \frac{-1}{-1} = 1$$

$$\frac{-1}{-1} + \frac{-1}{-1} = -2$$

$$0 = x(x-1)(x-1)(x-1)$$

$$\begin{array}{r} \downarrow \\ 1 \quad -3 \quad 3 \quad -1 \\ \underline{ \quad \quad \quad } \\ 1 \quad -2 \quad 1 \end{array}$$

$$x=0 \quad | \quad x-1=0$$

$$(0,0) \quad | \quad x=1$$

$$(0,0) \quad | \quad (1,0)$$

② y-int (x=0)

$$f(x) = x^4 - 3x^3 + 3x^2 - x$$

$$f(0) = (0)^4 - 3(0)^3 + 3(0)^2 - (0) = 0$$

$$(0,0)$$

Intervals of Inc/Dec:

$$f(x) = x^4 - 3x^3 + 3x^2 - x$$

$$f'(x) = 4x^3 - 9x^2 + 6x - 1 \text{ syn. sub.}$$

$$f'(x) = (x-1)(4x^2 - 5x + 1) \text{ Hard - Trinomial}$$

$$f'(x) = (x-1)(x-\frac{1}{4})(x-\frac{1}{4})$$

$$\frac{-1}{-1} x \frac{-4}{-4} = 4$$

$$\frac{-1}{-1} + \frac{-4}{-4} = -5$$

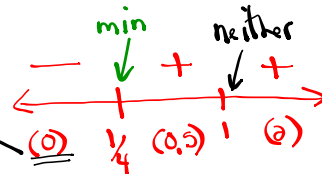
$$f'(x) = (x-1)(4x-1)(x-1)$$

$$f'(x) = (x-1)^2(4x-1)$$

$$4(1)^3 - 9(1)^2 + 6(1) - 1$$

$$4 - 9 + 6 - 1$$

$$\begin{array}{r} \downarrow \\ 4 \quad -9 \quad 6 \quad -1 \\ \underline{ \quad \quad \quad } \\ 4 \quad -5 \quad 1 \end{array}$$



$$\text{CV: } (x-1)^2 = 0 \quad | \quad 4x-1 = 0$$

$$x-1 = 0 \quad | \quad 4x = 1$$

$$x = 1 \quad | \quad x = \frac{1}{4}$$

Decreasing on $(-\infty, \frac{1}{4})$
 $x < \frac{1}{4}$

Increasing on $(\frac{1}{4}, \infty)$
 $x > \frac{1}{4}$

$$x = \frac{1}{4}, 1$$