

Questions from Homework

$$\textcircled{2} \quad f(3) = 33 \quad \boxed{f(x) = 4x^2 - 2x + 3}$$

$$f'(3) = 22 \quad f'(x) = 8x - 2$$

$$f''(3) = 8 \quad f''(x) = 8$$

$$\textcircled{4} \quad f(x) = \sqrt{1+x^3} = (1+x^3)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(1+x^3)^{-\frac{1}{2}}(3x^2) = \frac{3x^2}{2(1+x^3)^{\frac{1}{2}}}$$

$$f''(x) = \frac{6x(2)(1+x^3)^{-\frac{1}{2}} - 3x^2(1+x^3)^{-\frac{1}{2}}(3x^2)}{[2(1+x^3)^{\frac{1}{2}}]^2}$$

$$f''(x) = \frac{12x(1+x^3)^{-\frac{1}{2}} - 9x^4(1+x^3)^{-\frac{1}{2}}}{4(1+x^3)}$$

$$f''(x) = \frac{3x(1+x^3)^{-\frac{1}{2}} [4(1+x^3) - 3x^3]}{4(1+x^3)}$$

$$f''(x) = \frac{3x(4+x^3)}{4(1+x^3)^{\frac{3}{2}}} = \frac{3x(4+x^3)}{4\sqrt{(1+x^3)^3}}$$

$$f''(2) = \frac{6(12)}{4\sqrt{729}} = \frac{18}{27} = \left(\frac{2}{3}\right)$$

Implicit Differentiation

So far we have described functions by expressing one variable *explicitly* in terms of another variable: for example,

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

■ Sometimes an equation only implicitly defines y as a function (or functions) of x .

■ Examples

■ $x^2 + y^2 = 25$

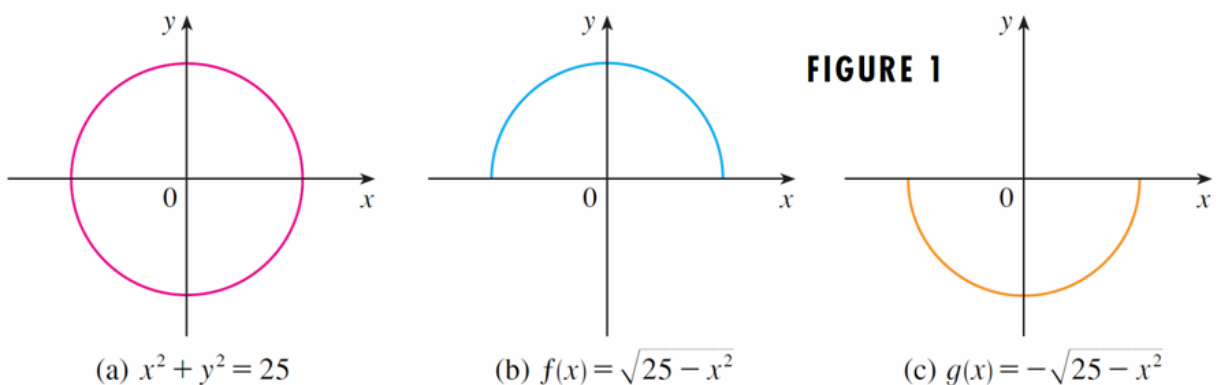
■ $x^3 + y^3 = 6xy$

$$\begin{aligned}
 x^2 + y^2 &= 25 \\
 y^2 &= 25 - x^2 \\
 y &= \pm \sqrt{25 - x^2}
 \end{aligned}$$

$(5-x)(5-x)$
 $25 - 5x - 5x + x^2$
 $25 - 10x + x^2$

- The first equation could easily be rearranged for $y = \dots$

$$y = \pm \sqrt{25 - x^2} \quad \leftarrow \text{Actually gives two functions}$$



Implicit Differentiation

- There is a way called *implicit differentiation* to find $\frac{dy}{dx}$ without solving for y :
- ① ■ First differentiate both sides of the equation with respect to x ;
- ② ■ Then solve the resulting equation for y' . $\left(\frac{dy}{dx}\right)$
- We will always assume that the given equation does indeed define y as a differentiable function of x .

Example

- For the circle $x^2 + y^2 = 25$, find
 - a) dy/dx
 - b) an equation of the tangent at the point $(3, 4)$.

Solution:

$$x_1 = 3 \quad y_1 = 4$$

Start by differentiating both sides of the equation:

$$a) \quad x^2 + y^2 = 25$$

$$2(x)(1) + 2(y)\left(\frac{dy}{dx}\right) = 0$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$2y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

b) Find the equation of the tangent @ $(3, 4)$

(i) Find m :

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{(3)}{(4)} = \left(-\frac{3}{4}\right) \leftarrow m$$

(ii) Find equation:

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -\frac{3}{4}(x - 3)$$

$$y - 4 = -\frac{3x}{4} + \frac{9}{4}$$

$$y = -\frac{3x}{4} + \frac{9}{4} + 4$$

$$\boxed{y = -\frac{3x}{4} + \frac{25}{4}} \quad \text{or} \quad 4y = -3x + 25$$

$$\boxed{3x + 4y - 25 = 0}$$

Same Example Revisited

- Since it is easy to solve this equation for y , we
 - do so, and then
 - find the equation of the tangent line by earlier methods, and then
 - compare the result with our preceding answer:

Solution

- Solving the equation gives $y = \pm\sqrt{25-x^2}$ as before.
- The point $(3, 4)$ lies on the upper semicircle $y = \sqrt{25-x^2}$ and so we consider the function $f(x) = \sqrt{25-x^2}$

Differentiate f :

$$(i) \quad y = (25-x^2)^{1/2}$$

$$y' = \frac{1}{2}(25-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{25-x^2}}$$

$$(ii) \quad y' = \frac{-x}{\sqrt{25-x^2}} = \frac{-(3)}{\sqrt{25-(3)^2}} = \left(\frac{-3}{4}\right) \rightarrow m$$

$$(iii) \quad y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{-3}{4}(x - 3)$$

$$y - 4 = \frac{-3x + 9}{4}$$

$$\boxed{y = \frac{-3x + 25}{4}}$$

Solution (cont'd)

- So $f'(3) = -\frac{3}{\sqrt{25-3^2}} = -\frac{3}{4}$,

leading to the same equation

$$3x + 4y = 25$$

for the tangent that we obtained earlier.

- Note that although this problem could be done both ways, implicit differentiation was easier!

Sometimes **Implicit Differentiation** is not only the easiest way, it's the *only* way

Example:

Given

$$x^3 + y^3 = (6xy)$$

f(x) g(x)

Find $\frac{dy}{dx}$

$$(3x^2) + 3y^2 \frac{dy}{dx} = 6y + 6x \frac{dy}{dx}$$

Product

$$3y^2 \frac{dy}{dx} - 6x \frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx} (3y^2 - 6x) = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

$$\frac{dy}{dx} = \frac{\cancel{3}(2y - x^2)}{\cancel{3}(y^2 - 2x)} = \boxed{\frac{2y - x^2}{y^2 - 2x}}$$

$$2x^5 + x^4y + y^5 = 36$$

Find $\frac{dy}{dx}$

Homework

Exercise 2.7 Page 107
Do # 1-3, 5, 7